Resolution of linear systems by the Gauss method

Exercise 1 Decide, for the following systems of equations with unknown variables x_1, x_2, ..., x_n and parameters s, t, if they are linear :

$$a)\left\{\begin{array}{lll} x_1\sin(t) & +x_2 & =3 \\ x_1e^t & +3x_2 & =t^2 \end{array}\right. \qquad b)\left\{\begin{array}{lllll} \frac{1}{x_1} & +\frac{2}{x_2} & +\dots & +\frac{n}{x_n} & =n! \\ x_1 & +\frac{x_2}{2} & +\dots & +\frac{x_n}{n} & =\frac{1}{n!} \end{array}\right.$$

$$c)\left\{\begin{array}{ccc} \sqrt{(x_1+sx_2+t)^2-4sx_2(x_1+t)} & = & 0 \\ x_1\ln s-\pi x_2+e^t x_n & = & 0 \end{array}\right. \quad d)\left\{\begin{array}{ccc} (1+sx_1)(3+tx_2)-(2+tx_1)(5+sx_2) & = & 8 \\ (x_3+s)^2-(x_3-s)^2+x_2 & = & 0 \end{array}\right.$$

Solution of exercise 1:
Recall that a system of equations is called *linear* if every equation in the system is linear. An equation with unknown variables x_1, x_2, ..., x_n is linear if it is of the form $\sum_{i=1}^n a_ix_i = b$, where a_i and b are constants which can depend (possibly non linearly!) on some parameters (but which are independant of the variables).

a) This system is linear.
b) The first equation is not linear, hence the system is not linear.
c) One has $\sqrt{(x_1+sx_2+t)^2-4sx_2(x_1+t)} = \sqrt{(x_1-sx_2+t)^2} = \pm(x_1-sx_2+t)$ depending on the sign of the expression (x_1-sx_2+t). However, since $|u|=0 \Leftrightarrow u=0$, the first equation is equivalent to $x_1-sx_2+t=0 \Leftrightarrow x_1-sx_2=-t$ which is linear in x_1, x_2 (remark that this would not be the case if the right hand side of the first equation was non-zero). The second equation is also linear in x_1, x_2, x_n. Hence the system is linear.
d) After simplification, one gets: $\left\{\begin{array}{ccc} x_1(3s-5t)+x_2(t-2s) & = & 15 \\ x_2+4sx_3 & = & 0 \end{array}\right.$ which is linear.

Exercise 2 Using Gauss's algorithm, solve the following system:

$$\mathcal{S}:\left\{\begin{array}{ccccccccc} 2x_1 & + & 4x_2 & - & 6x_3 & - & 2x_4 & = & 2 \\ 3x_1 & + & 6x_2 & - & 7x_3 & + & 4x_4 & = & 2 \\ 5x_1 & + & 10x_2 & - & 11x_3 & + & 6x_4 & = & 3 \end{array}\right. .$$

Solution of exercise 2:
One has the following equivalences:

$$\mathcal{S}\Leftrightarrow\left\{\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & - & x_4 & = & 1 \\ 3x_1 & + & 6x_2 & - & 7x_3 & + & 4x_4 & = & 2 \\ 5x_1 & + & 10x_2 & - & 11x_3 & + & 6x_4 & = & 3 \end{array}\right. \quad L_1 \leftarrow \tfrac{1}{2}L_1$$

$$\Leftrightarrow\left\{\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & - & x_4 & = & 1 \\ & & & & 2x_3 & + & 7x_4 & = & -1 \\ & & & & 4x_3 & + & 11x_4 & = & -2 \end{array}\right. \quad \begin{array}{l} \\ L_2 \leftarrow L_2-3L_1 \\ L_3 \leftarrow L_3-5L_1 \end{array}$$

$$\Leftrightarrow\left\{\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & - & x_4 & = & 1 \\ & & & & 2x_3 & + & 7x_4 & = & -1 \\ & & & & & - & 3x_4 & = & 0 \end{array}\right. \quad \begin{array}{l} \\ \\ L_3 \leftarrow L_3-2L_2 \end{array}$$

$$\Leftrightarrow\left\{\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & - & x_4 & = & 1 \\ & & & & 2x_3 & + & 7x_4 & = & -1 \\ & & & & & & x_4 & = & 0 \end{array}\right. \quad \begin{array}{l} \\ \\ L_3 \leftarrow -\tfrac{1}{3}L_3 \end{array}$$

$$\Leftrightarrow \left\{\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & - & x_4 & = & 1 \\ & & & & 2x_3 & & & = & -1 \\ & & & & & & x_4 & = & 0 \end{array}\right. \quad L_2 \leftarrow L_2 - 7L_3$$

$$\Leftrightarrow \left\{\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & - & x_4 & = & 1 \\ & & & & x_3 & & & = & -\frac{1}{2} \\ & & & & & & x_4 & = & 0 \end{array}\right. \quad L_2 \leftarrow \frac{1}{2}L_2$$

$$\Leftrightarrow \left\{\begin{array}{ccccccc} x_1 & + & 2x_2 & & & = & -\frac{1}{2} \\ & & & x_3 & & = & -\frac{1}{2} \\ & & & & x_4 & = & 0 \end{array}\right. \quad L_1 \leftarrow L_1 + 3L_2 + L_3$$

It follows that the set of solutions of the system $\mathcal{S}$ is the set of vectors

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - 2\lambda \\ \lambda \\ -\frac{1}{2} \\ 0 \end{pmatrix}.$$

Exercise 3 Solve the following system:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ -1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 3 \\ 1 & -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

Solution of exercise 3
This is a system of 5 equations relating 4 unknowns. Therefore there will appear $5 - 4 = 1$ conditions of compatibility for the system to have a solution. The extended matrix of the system is the following:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ -1 & 2 & 0 & 1 & 2 \\ 3 & 2 & 1 & 2 & -1 \\ 4 & 3 & 2 & 3 & 3 \\ 1 & -1 & 1 & -2 & 1 \end{array}\right)$$

The Gauss algorithm applied to the system gives the following succession of elementary operations:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 3 & 3 \\ 0 & 2 & -2 & -4 & -4 \\ 0 & 3 & -2 & -5 & -1 \\ 0 & -1 & 0 & -4 & 0 \end{array}\right) \quad \begin{array}{l} \\ L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1 \\ L_5 \leftarrow L_5 - L_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 3 & -2 & -5 & -1 \\ 0 & 2 & 1 & 3 & 3 \end{array}\right) \quad \begin{array}{l} \\ L_2 \leftarrow L_5 \quad \text{then} \quad L_2 \leftarrow -L_2 \\ L_3 \leftarrow \frac{-1}{2}L_3 \\ \\ L_5 \leftarrow L_2 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & -2 & -17 & -1 \\ 0 & 0 & 1 & -5 & 3 \end{array}\right) \quad \begin{array}{l} \\ \\ L_3 \leftarrow L_3 + L_2 \\ L_4 \leftarrow L_4 - 3L_2 \\ L_5 \leftarrow L_5 - 2L_2 \end{array} \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & -11 & 1 \end{array}\right) \quad \begin{array}{l} \\ \\ \\ L_4 \leftarrow L_4 + 2L_3 \\ L_5 \leftarrow L_5 - L_3 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & 0 & \frac{-28}{5} \end{array}\right) \quad L_5 \leftarrow L_5 - \frac{11}{5}L_4$$

Since $\frac{-28}{5} \neq 0$, the system has no solution.

Exercise 4 Resolve the following system:

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & 3 & -1 & 2 \\ -2 & -1 & 1 & -3 & 2 \\ 3 & 2 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Solution of exercise 4:
This is a system of 4 equations with 5 unknowns. Therefore we expect the system to have a line of solutions. We apply the Gauss algorithm to the extended matrix of the system:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 2 & 1 & 3 & -1 & 2 & 0 \\ -2 & -1 & 1 & -3 & 2 & 1 \\ 3 & 2 & 0 & 1 & -1 & 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & -3 & 0 & -4 \\ 0 & -1 & 5 & -1 & 4 & 5 \\ 0 & 2 & -6 & -2 & -4 & -5 \end{array}\right) \quad \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 + 2L_1 \\ L_4 \leftarrow L_4 - 3L_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & -3 & 0 & -4 \\ 0 & 0 & 4 & -4 & 4 & 1 \\ 0 & 0 & -4 & 4 & -4 & 3 \end{array}\right) \quad \begin{array}{l} L_3 \leftarrow L_3 + L_2 \\ L_4 \leftarrow L_4 - 2L_2 \end{array} \quad \Leftrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & -3 & 0 & -4 \\ 0 & 0 & 4 & -4 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right) \quad L_4 \leftarrow L_4 + L_3$$

Since $4 \neq 0$, the system has in fact no solution (contrary to what we expected...).

Exercise 5 Let a be a real number. Consider the following linear system:

$$\mathcal{S}_a : \left\{ \begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 2 \\ x & + & 3y & - & 2z & = & 5 \\ 2x & - & y & + & az & = & 1 \end{array} \right.$$

1. According to the values of the parameter a, determine if the system $\mathcal{S}_a$ can:
   (i) admit no solution ;
   (ii) admit a unique solution ;
   (iii) admit infinitely many solutions.
2. Solve the system $\mathcal{S}_a$ when it has one or more solutions.

Solution of exercise 5:

$$\mathcal{S}_a \Leftrightarrow \left\{ \begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 2 \\ & & 5y & - & 5z & = & 3 \\ & & 3y & + & (a-6)z & = & -3 \end{array} \right. \quad \begin{array}{l} \\ L_2 - L_1 \\ L_3 - 2L_1 \end{array}$$

$$\Leftrightarrow \left\{ \begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 2 \\ & & 5y & - & 5z & = & 3 \\ & & & & (5a-15)z & = & -24 \end{array} \right. \quad \begin{array}{l} \\ \\ 5L_3 - 3L_2 \end{array}$$

1. (i) For $a = 3$ the system $\mathcal{S}_a$ has no solution.

(ii) For $a \neq 3$, the system admits a unique solution.

(iii) There is no value of the parameter a for which the system admits infinitely many solutions.

2. For $a \neq 3$ the system is equivalent to:

$$\begin{cases} x & = 2+2\frac{3}{5}-2\frac{24}{(5a-15)}+3\frac{24}{(5a-15)} \\ \quad y & = \frac{3}{5}-\frac{24}{(5a-15)} \\ \quad\quad z & = -\frac{24}{(5a-15)} \end{cases}$$

and has the following unique solution:

$$\vec{v} = \begin{pmatrix} \frac{16}{5}+\frac{24}{(5a-15)} \\ \frac{3}{5}-\frac{24}{(5a-15)} \\ -\frac{24}{(5a-15)} \end{pmatrix}$$

Exercise 6 The complex vectors (z, w) et (z', w') are related by the formula $(z', w') = (z+iw, (1+i)z + (1-2i)w)$. A student who does not like complex numbers set $z = x+iy$, $w = u+iv$, $z' = x'+iy'$ and $w' = u'+iv'$.

1. Express (x', y', u', v') as a function of (x, y, u, v).
2. Resolve the system $(x', y', u', v') = (1, 2, 3, 4)$.

Solution of exercise 6:

1. One obtains the following system:

$$\begin{cases} x' = x - v \\ y' = y + u \\ u' = x - y + u + 2v \\ v' = x + y - 2u + v \end{cases}$$

2. The equation $(x', y', u', v') = (1, 2, 3, 4)$ gives rise to a system of 4 equations with 4 unknowns (x, y, u, v) whose extended matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & -1 & 1 & 2 & 3 \\ 1 & 1 & -2 & 1 & 4 \end{array}\right) \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 1 & -2 & 2 & 3 \end{array}\right) \begin{array}{l} \\ \\ L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & -3 & 2 & 1 \end{array}\right) \begin{array}{l} \\ \\ L_3 \leftarrow L_3 + L_2 \\ L_4 \leftarrow L_4 - L_2 \end{array} \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & \frac{13}{2} & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ L_4 \leftarrow L_4 + \frac{3}{2}L_3 \end{array}$$

One has: $v = \frac{14}{13}$, $2u = 4 - 3\frac{14}{13}$ hence $u = 2 - \frac{21}{13} = \frac{5}{13}$, $y = 2 - \frac{5}{13} = \frac{21}{13}$ and $x = 1 + \frac{14}{13} = \frac{27}{13}$. In conclusion, the solution of the equation $(x', y', u', v') = (1, 2, 3, 4)$ is the 4-uple $(x, y, u, v) = (\frac{27}{13}, \frac{21}{13}, \frac{5}{13}, \frac{14}{13})$.

Exercise 7 A biker practices every Sunday by riding from Issy to Labat and back. The road is not flat, but comprises uphill, downhill, and flat sections. The biker covers 15 kilometers per hour uphill, 20km/h on flat ground, and 30km/h downhill. He needs two hours from Issy to Labat, and three hours on his way back. The steepness of the slopes is everywhere 5% (away from the flat sections).

1. How far is Issy from Labat, which town lies higher, and by how much?

2. A second, stronger biker achieves 20km/h uphill, 30km/h on flat, and 40km/h downhill. He needs only 3h40 for the round trip. Determine all three lengths (of the uphill, downhill, and flat sections).

Solution of exercise 7:
On the journey from Issy to Labat, let x be the length of the uphill section, y the length of the flat section, and z the length of the downhill section (in kilometers). The data for the first biker translates as

$$\begin{cases} (L_1): & x/15 + y/20 + z/30 &= 2 \\ (L_2): & x/30 + y/20 + z/15 &= 3\ . \end{cases}$$

1. The addition $(L_1 + L_2)$ yields $\frac{x}{10} + \frac{y}{10} + \frac{z}{10} = 5$ i.e. $x + y + z = 50$. The road distance between Issy and Labat is thus 50km.

   The subtraction $(L_2 - L_1)$ yields $\frac{-x}{30} + \frac{z}{30} = 1$ i.e. $z - x = 30$. The downhill section is 30km longer than the uphill section. Therefore Labat lies $\frac{5}{100} \cdot 30 = 1.5$km (or 1500m) lower.

2. The third biker gives

$$(L_3): \quad x\left(\frac{1}{20} + \frac{1}{40}\right) + y\left(\frac{1}{30} + \frac{1}{30}\right) + z\left(\frac{1}{40} + \frac{1}{20}\right) = 3 + \frac{40}{60}\ ,$$

   which can be written

$$(L_3): \quad x \cdot \frac{3}{40} + y \cdot \frac{1}{15} + z \cdot \frac{3}{40} = \frac{11}{3}\ .$$

   Computing $(L_1 + L_2 - \frac{4}{3}L_3)$ to eliminate x and z, we get:

$$y\left(\frac{1}{10} - \frac{4}{3} \cdot \frac{1}{15}\right) = 5 - \frac{4}{3} \cdot \frac{11}{3}$$

   i.e. $\frac{y}{90} = \frac{1}{9}$: therefore $y = 10$.

   Using the first question, we now have $x + z = 50 - 10 = 40$ and $z - x = 30$: the half-sum and half-difference of these two identities yield $x = 5$ and $z = 35$.

Exercise 8 Let a, b, and c be three real numbers.

1. Which relationship should the parameters a, b and c satisfy for the following system to have at least one solution?

$$\mathcal{S}_{abc}: \begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c \end{cases}$$

2. Can the system $\mathcal{S}_{abc}$ have a unique solution in $\mathbb{R}^3$?

Solution of exercise 8:

1. One has:

$$\mathcal{S}_{abc} \Leftrightarrow \begin{cases} x + 2y - 3z = a & \\ \quad 2y - 5z = b - 2a & L_2 - 2L_1 \\ \quad -4y + 10z = c - a & L_3 - L_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + 2y - 3z = a & \\ \quad 2y - 5z = b - 2a & \\ \quad 0 = c + 2b - 5a & L_3 + 2L_2 \end{cases}\ .$$

   Therefore the system S_{abc} admits a solution if and only if the following relationship is satisfied: $c + 2b - 5a = 0$.

2. Under the condition $c+2b-5a=0$, the set of solutions of the system $\mathcal{S}_{abc}$ is the set of vectors $\vec{v}$ of the form:

$$\vec{v}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3a-b-2\lambda \\ \frac{1}{2}(b-2a+5\lambda) \\ \lambda \end{pmatrix}=\begin{pmatrix} 3a-b \\ \frac{b}{2}-a \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} -2 \\ \frac{5}{2} \\ 1 \end{pmatrix},$$

where $\lambda \in \mathbb{R}$ is arbitrary. Consequently, there is no way for the system $\mathcal{S}_{abc}$ to admit a unique solution in $\mathbb{R}^3$: either $c+2b-5a \neq 0$ and it has no solution at all, or $c+2b-5a=0$ and it has infinitely many solutions given by the affine line passing through the point $(3a-b, \frac{b}{2}-a, 0)^T$, and directed by the vector $(-2, \frac{5}{2}, 1)^T$.

Exercise 9 Find, according to the values of the parameter m, the set of solutions of the following systems:

$$\mathcal{S}_1 : \begin{cases} x + (m+1)y = m+2 \\ mx + (m+4)y = 3 \end{cases} \qquad \mathcal{S}_2 : \begin{cases} mx + (m-1)y = m+2 \\ (m+1)x - my = 5m+3 \end{cases}$$

Solution of exercise 9:

$$\mathcal{S}_1 \Leftrightarrow \begin{cases} x + (m+1)y = m+2 \\ (-m^2+4)y = 3-2m-m^2 \quad L_2 - mL_1 \end{cases}$$

- If $m^2=4$, then the second equation reads $0=-1-2m$ which is impossible for $m=\pm 2$. Hence for $m^2=4$ the system has no solution.
- If $m^2 \neq 4$, the system has a unique solution $\vec{v}=\begin{pmatrix} (m+2)-(m+1)\frac{3-2m-m^2}{4-m^2} \\ \frac{3-2m-m^2}{4-m^2} \end{pmatrix}$.

For $\mathcal{S}_2$,

- if $m=0$, then the system has a unique solution $\vec{v}=\begin{pmatrix} 3 \\ -2 \end{pmatrix}$;
- if $m \neq 0$ and $m \neq -1$, the system is equivalent to:

$$\mathcal{S}_2 \Leftrightarrow \begin{cases} mx + (m-1)y = m+2 \\ (-2m^2+1)y = -2(-2m^2+1) \quad [mL_2-(m+1)L_1] \end{cases}$$

  - In the case where $m^2 \neq \frac{1}{2}$, the system has also a unique solution $\vec{v}=\begin{pmatrix} 3 \\ -2 \end{pmatrix}$;
  - if $m^2=\frac{1}{2}$ then the system has infinitely many solutions lying on the line in $\mathbb{R}^2$ whose cartesian equation is $mx+(m-1)y=m+2$.
    * For $m=\frac{1}{\sqrt{2}}$, this equation is equivalent to $x+(1-\sqrt{2})y=1+2\sqrt{2}$, and the set of solutions is the set of vectors $\vec{v}\in\mathbb{R}^2$ of the form $\vec{v}=\begin{pmatrix} 1+2\sqrt{2} \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} \sqrt{2}-1 \\ 1 \end{pmatrix}$, where $\lambda\in\mathbb{R}$.
    * For $m=-\frac{1}{\sqrt{2}}$, this equation is equivalent to $x+(1+\sqrt{2})y=1-2\sqrt{2}$, and the set of solutions is the set of vectors $\vec{v}\in\mathbb{R}^2$ of the form $\vec{v}=\begin{pmatrix} 1-2\sqrt{2} \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} -\sqrt{2}-1 \\ 1 \end{pmatrix}$, where $\lambda\in\mathbb{R}$.

The conclusion is the following:

- for $m^2 \neq \frac{1}{2}$, the system $\mathcal{S}_2$ has a unique solution, the vector $\vec{v}=\begin{pmatrix} 3 \\ -2 \end{pmatrix}$;
- for $m=\frac{1}{\sqrt{2}}$, the system has infinitely many solutions given by $\vec{v}=\begin{pmatrix} 1+2\sqrt{2} \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} \sqrt{2}-1 \\ 1 \end{pmatrix}$, where $\lambda\in\mathbb{R}$;

- for $m = -\frac{1}{\sqrt{2}}$, the system has infinitely many solutions given by $\vec{v} \in \mathbb{R}^2$ of the form $\vec{v} = \begin{pmatrix} 1 - 2\sqrt{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\sqrt{2} - 1 \\ 1 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.