University of Lille I
PC first year

list of exercises n° 3

Linear maps – subvectorspaces of $\mathbb{R}^n$

Exercise 1 1. Endow $\mathbb{R}^2$ with an orthonormal frame $(O, \vec{i}, \vec{j})$. Show that a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is uniquely determined by its values on the vectors $\vec{i}$ and $\vec{j}$.

2. In the basis $\{\vec{i}, \vec{j}\}$, what is the matrix of the orthogonal symmetry with respect to the horizontal axis?

3. In the basis $\{\vec{i}, \vec{j}\}$, what is the matrix of the orthogonal projection to the horizontal axis?

4. In the basis $\{\vec{i}, \vec{j}\}$, what is the matrix of the rotation of angle θ and center O?

5. In the basis $\{\vec{i}, \vec{j}\}$, what is the matrix of the homothety of center O and ratio k?

6. In the basis $\{\vec{i}, \vec{j}\}$, what is the matrix of the symmetry of center O?

7. Is a translation a linear map?

Solution of Exercise 1 :

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Consider any vector $\vec{v}$ in $\mathbb{R}^2$. Since $\{\vec{i}, \vec{j}\}$ is a basis of $\mathbb{R}^2$, $\vec{v}$ can be uniquely written as : $\vec{v} = x\vec{i} + y\vec{j}$. By linearity of f, one has : $f(\vec{v}) = f(x\vec{i} + y\vec{j}) = xf(\vec{i}) + yf(\vec{j})$. Therefore the values of f on the vectors $\vec{i}$ and $\vec{j}$, determine the value of f on any vector of $\mathbb{R}^2$. Two linear maps taking the same values on $\vec{i}$ and $\vec{j}$ will coincide on $\mathbb{R}^2$.

2. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the orthogonal symmetry with respect to the horizontal axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the orthogonal projection to the horizontal axis is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

4. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the rotation of angle θ and center O is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

5. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the homothety of center O and ratio k is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

6. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the symmetry of center O is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

7. A linear map f from $\mathbb{R}^n$ into $\mathbb{R}^p$ necessarily maps $\vec{0} \in \mathbb{R}^n$ onto $\vec{0} \in \mathbb{R}^p$. The translation by a given vector $\vec{u} \in \mathbb{R}^2$ takes $\vec{v} \in \mathbb{R}^2$ to $\vec{v} + \vec{u} \in \mathbb{R}^2$. In particular, the translation of vector $\vec{u}$ takes $\vec{0} \in \mathbb{R}^2$ to $\vec{u} \in \mathbb{R}^2$. Therefore, if $\vec{u} \neq \vec{0}$, the translation of vector $\vec{u}$ is not a linear map.

Exercise 2 Let f be the map from $\mathbb{R}^4$ to $\mathbb{R}^4$ defined by:

$$f(x, y, z, t) = \left(\ x + y + z + t, x + y + z + t, x + y + z + t, 2x + 2y + 2z + 2t \ \right).$$

1. Show that f is linear and write down its matrix in the canonical basis of $\mathbb{R}^4$.

2. Check that the vectors $\vec{a} = (1, -1, 0, 0)$, $\vec{b} = (0, 1, -1, 0)$ and $\vec{c} = (0, 0, 1, -1)$ all belong to $\ker f$.

3. Check that the vector $\vec{d} = (5, 5, 5, 10)$ belongs to Imf.

Solution of Exercise 2:

1. One has to verify that, for any vectors $\vec{v_1}$ and $\vec{v_2}$ in $\mathbb{R}^4$ and any $\lambda \in \mathbb{R}$, one has $f(\vec{v_1} + \lambda\vec{v_2}) = f(\vec{v_1}) + \lambda f(\vec{v_2})$. Denote by (x_1, y_1, z_1, t_1) (resp. (x_2, y_2, z_2, t_2)) the coordinates of the vector $\vec{v_1}$ (resp. $\vec{v_2}$) in the canonical basis of $\mathbb{R}^4$. The coordinates of the vector $\vec{v_1} + \lambda\vec{v_2}$ are $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2)$. Therefore, using the formula that defines the map f, one has:

$$\begin{aligned} f(\vec{v_1} + \lambda\vec{v_2}) &= f(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2) \\ &= \begin{pmatrix} x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\ x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\ x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\ 2(x_1 + \lambda x_2) + 2(y_1 + \lambda y_2) + 2(z_1 + \lambda z_2) + 2(t_1 + \lambda t_2) \end{pmatrix}. \end{aligned}$$

On the other hand,

$$f(\vec{v_1}) = \begin{pmatrix} x_1 + y_1 + z_1 + t_1 \\ x_1 + y_1 + z_1 + t_1 \\ x_1 + y_1 + z_1 + t_1 \\ 2x_1 + 2y_1 + 2z_1 + 2t_1 \end{pmatrix} ; \quad f(\vec{v_2}) = \begin{pmatrix} x_2 + y_2 + z_2 + t_2 \\ x_2 + y_2 + z_2 + t_2 \\ x_2 + y_2 + z_2 + t_2 \\ 2x_2 + 2y_2 + 2z_2 + 2t_2 \end{pmatrix} ;$$

$$\lambda f(\vec{v_2}) = \begin{pmatrix} \lambda(x_2 + y_2 + z_2 + t_2) \\ \lambda(x_2 + y_2 + z_2 + t_2) \\ \lambda(x_2 + y_2 + z_2 + t_2) \\ \lambda(2x_2 + 2y_2 + 2z_2 + 2t_2) \end{pmatrix} ;$$

and

$$f(\vec{v_1}) + \lambda f(\vec{v_2}) = \begin{pmatrix} x_1 + y_1 + z_1 + t_1 + \lambda(x_2 + y_2 + z_2 + t_2) \\ x_1 + y_1 + z_1 + t_1 + \lambda(x_2 + y_2 + z_2 + t_2) \\ x_1 + y_1 + z_1 + t_1 + \lambda(x_2 + y_2 + z_2 + t_2) \\ 2x_2 + 2y_2 + 2z_2 + 2t_2 + \lambda(2x_2 + 2y_2 + 2z_2 + 2t_2) \end{pmatrix}$$

By commutativity of the reals, one obtains $f(\vec{v_1} + \lambda\vec{v_2}) = f(\vec{v_1}) + \lambda f(\vec{v_2})$.

The matrix of f in the canonical basis of $\mathbb{R}^4$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

2. Let us compute the images of the vectors $\vec{a} = (1, -1, 0, 0)$, $\vec{b} = (0, 1, -1, 0)$ and $\vec{c} = (0, 0, 1, -1)$. One has

$$f(\vec{a}) = f(1, -1, 0, 0) = (1 - 1, 1 - 1, 1 - 1, 2 - 2) = (0, 0, 0, 0);$$
$$f(\vec{b}) = f(0, 1, -1, 0) = (1 - 1, 1 - 1, 1 - 1, 2 - 2) = (0, 0, 0, 0);$$
$$f(\vec{c}) = f(0, 0, 1, -1) = (1 - 1, 1 - 1, 1 - 1, 2 - 2) = (0, 0, 0, 0).$$

Therefore $\vec{a}$, $\vec{b}$ and $\vec{c}$ belong to ker$f$.

3. Since the vector $\vec{d} = (5, 5, 5, 10)$ is the image of the vector $(5, 0, 0, 0)$, $\vec{d}$ belongs to Imf.

Exercise 3 Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$f(x, y, z) = (x + 2y + z, 2x + y + 3z, -x - y - z).$$

1. Justify that f is linear.

2. Give the matrix of f in the canonical basis of $\mathbb{R}^3$.

3. (a) Determine a basis and the dimension of the kernel of f, denoted by ker$f$.

   (b) Is the map f injective?

4. (a) Give the rank of f and a basis of Imf.

   (b) Is the map f surjective?

Solution of Exercise 3:

1. One has to verify that, for any vectors $\vec{v_1}$ and $\vec{v_2}$ in $\mathbb{R}^3$ and any $\lambda \in \mathbb{R}$, one has $f(\vec{v_1} + \lambda\vec{v_2}) = f(\vec{v_1}) + \lambda f(\vec{v_2})$. It is the same kind of computation as in Exercise 2, question 1.

2. The matrix of f in the canonical basis of $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix}.$$

3. (a) The kernel of f, written $\ker f$, is the set of vectors which are mapped onto $\vec{0}$ by f. Therefore, a vector $\vec{v} = (x, y, z) \in \mathbb{R}^3$ belongs to ker$f$ if and only if (x, y, z) is a solution of the following system:

$$\begin{cases} x + 2y + z = 0 \\ 2x + y + 3z = 0 \\ -x - y - z = 0 \end{cases}$$

Applying the Gauss method, one obtains that the above system is equivalent to

$$\Leftrightarrow \begin{cases} x + 2y + z = 0 \\ -3y + 2z = 0 \\ -3y - 2z = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y + z = 0 \\ -3y + 2z = 0 \\ 4z = 0 \end{cases}.$$

Therefore the unique solution of the system is $\vec{v} = \vec{0}$, and ker$f = \{\vec{0}\}$. The dimension of ker$f$ is therefore 0. The empty set $\emptyset$ is a basis of ker$f$.

   (b) For a linear map, being injective is equivalent to ker$f = \{\vec{0}\}$. Hence, by the previous question, f is injective.

4. (a) There are many ways to answer this question. Recall that a vector $\vec{b} \in \mathbb{R}^3$ belongs to Imf if and only if there exists $\vec{v} = (x, y, z) \in \mathbb{R}^3$ such that $f(\vec{v}) = \vec{b}$, or equivalently if $\vec{b}$ is a linear combination of the columns of the matrix associated to f. According to the expression of the matrix associated to f given in question 2., Imf is the vector space generated by the vectors $C_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $C_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

To find a basis of Imf, one way is to apply Gauss algorithm to the matrix of f in order to trigonalize it. One finds:

$$\begin{matrix} C_1 & C_2 & C_3 \end{matrix} \qquad\qquad \begin{matrix} C_2 \leftarrow C_2 - 2C_1 \\ C_3 \leftarrow C_3 - C_1 \end{matrix} \qquad\qquad C_3 \leftarrow 3C_3 + C_2$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Since the vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are column vectors of a triangular matrix, they are linearly independent. Since we applied the Gauss algorithm to the *columns* of the

matrix associated to f, they generate Imf. Consequently they form a basis of Imf which is therefore equal to $\mathbb{R}^3$.
Another way to find a basis of Imf, is to compute the determinant of the matrix associated to f. Since

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix} \neq 0,$$

the columns of this matrix are linearly independent. Therefore they form a basis of Imf.
A shorter way to answer this question, is to use Rank Theorem. Since f is an injective map from $\mathbb{R}^3$ into $\mathbb{R}^3$, one has

$$\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Leftrightarrow 3 = 0 + \dim \operatorname{Im} f.$$

Hence Im$f = \mathbb{R}^3$ since the only subspace of dimension 3 of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself. One concludes that the rank of f (which is by definition the dimension of Imf) is 3, and a basis of Imf is given, for example, by $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(b) Recall that a map $f \ : E \to F$ is surjective if and only if Im$f = F$. For a linear map, this is equivalent to dim Imf = dim F. By the previous question, the map considered in this exercise is surjective.

Exercise 4 1. Let f be a surjective linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$. What is the dimension of the kernel of f?

2. Let g be an injective map from $\mathbb{R}^{26}$ to $\mathbb{R}^{100}$. What is the dimension of the image of g?

3. Can there be a bijective linear map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$?

Solution of Exercise 4:

1. By the Rank Theorem, $\dim \ker f = \dim \mathbb{R}^4 - \dim \operatorname{Im} f$. Since f is supposed to be surjective, $\dim \operatorname{Im} f = 2$. Therefore $\dim \ker f = 4 - 2 = 2$.

2. By the Rank Theorem, $\dim \operatorname{Im} g = \dim \mathbb{R}^{26} - \dim \ker g$. Since g is supposed to be injective, $\dim \ker g = 0$. Hence $\dim \operatorname{Im} g = 26$.

3. By the Rank Theorem, an injective map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ satisfies $\dim \operatorname{Im} f = 50$. On the other hand, a surjective map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ satisfies $\dim \operatorname{Im} f = 72$. Consequently a map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ can not be injective *and* surjective. Therefore there exists no bijective map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$.

Exercise 5 Consider the matrix

$$A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix}.$$

1. Compute a basis of the kernel of A.

2. Compute a basis of the image of A.

Solution of Exercise 5: We will answer both questions at the same time. To do so, we will apply the Gauss algorithm to the columns of the matrix A and I_3 simultanously (here I_3 denotes the identity matrix of size $(3,3)$ having the same number of columns as A).

$$\begin{array}{cc} A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix} \\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} C_2 \leftarrow 2C_2 - 7C_1 \\ C_3 \leftarrow 2C_3 - C_1 \\ \begin{pmatrix} 2 & 0 & 0 \\ -1 & 11 & 1 \\ 3 & -11 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & -7 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} C_3 \leftarrow 11C_3 - C_2 \\ \begin{pmatrix} 2 & 0 & 0 \\ -1 & 11 & 0 \\ 3 & -11 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -7 & -4 \\ 0 & 2 & -2 \\ 0 & 0 & 22 \end{pmatrix} \end{array}.$$

It follows that a basis of Im A is given by the vectors $\vec{v_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 0 \\ 11 \\ -11 \end{pmatrix}$. Indeed, the columns of the upper matrix still generate Im A since we obtained them by applying the Gauss algorithm to the *columns* of A. The third column of the upper matrix being equal to the null vector, we only consider the first two columns, namely the vectors $\vec{v_1}$ and $\vec{v_2}$. These two vectors are linearly independant since they are two columns of a triangular matrix.

On the other hand the kernel of A is generated by the vector $\vec{u} = \begin{pmatrix} -4 \\ -2 \\ 22 \end{pmatrix}$. Indeed, by the Rank Theorem $\dim \ker f = \dim \mathbb{R}^3 - \dim \text{Im } f = 1$ since $\dim \text{Im } f = 2$. Moreover, one can verify that $\vec{u}$ is a non-zero vector of $\ker f$ by:

$$A\,\vec{u} = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ -2 \\ 22 \end{pmatrix} = \begin{pmatrix} -8 - 14 + 22 = 0 \\ 4 - 4 = 0 \\ -12 - 10 + 22 = 0 \end{pmatrix}.$$

Exercise 6 Consider the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & -1 & -3 \\ -3 & 5 & 2 & -3 \end{pmatrix}.$$

1. Compute a basis of the kernel of B.
2. Compute a basis of the image of B.

Solution of Exercise 6: We use the same technique as in the previous exercise: the Gauss algorithm on the columns of the matrix B and I_4 simultaneously (here I_4 denotes the identity matrix having as many columns as B, namely 4 columns).

$$\begin{array}{c} B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & -1 & -3 \\ -3 & 5 & 2 & -3 \end{pmatrix} \\ I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} C_2 \leftarrow C_2 - 2C_1 \\ C_3 \leftarrow C_3 - 3C_1 \\ C_4 - C_1 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 2 & -2 \\ -3 & 11 & 11 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} C_3 \leftarrow 2C_3 - C_2 \\ C_4 \leftarrow 2C_4 + C_2 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -3 & 11 & 11 & 11 \end{pmatrix} \\ \begin{pmatrix} 1 & -2 & -4 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} C_4 \leftarrow C_4 - C_3 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -3 & 11 & 11 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

Consequently, a basis of Im f is given by the three vectors $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 0 \\ 4 \\ 11 \end{pmatrix}$ and $\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$.

A basis of ker f is given by the vector $\vec{u} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 2 \end{pmatrix}$.

Exercise 7 Consider the matrix

$$C = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & -1 \\ 2 & 4 & 0 \\ 1 & 7 & 1 \end{pmatrix}.$$

1. Compute a basis of the kernel of C.

2. Compute a basis of the image of C.

Solution of Exercise 7: One has

$$\begin{matrix} C = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & -1 \\ 2 & 4 & 0 \\ 1 & 7 & 1 \end{pmatrix} \\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \xrightarrow{\substack{C_2 \leftarrow C_2 + 3C_1 \\ C_3 \leftarrow C_3 + C_1}} \begin{matrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 5 & 1 \\ 2 & 5 & 1 \\ 2 & 10 & 2 \\ 1 & 10 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \xrightarrow{C_3 \leftarrow 5C_3 - C_2} \begin{matrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 5 & 0 \\ 2 & 5 & 0 \\ 2 & 10 & 0 \\ 1 & 10 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{pmatrix} \end{matrix}.$$

Consequently a basis of Im f is given by the two vectors $\vec{v_1} = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 0 \\ 5 \\ 5 \\ 10 \\ 10 \end{pmatrix}$. A basis of ker f is given by the vector $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.