University of Lille I
PC first year **list of exercises n° 4**

Matrix calculus

Exercise 1 Let us consider the following matrices :

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 4 \\ 1 & 0 & -2 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}; \quad D = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}; \quad E = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}.$$

Compute, *when they make sense*, the following products : AB, BA, AC, CA, AD, AE, BC, BD, BE, CD, DE.

Solution of Exercise 1 :
Let us first recall that a product AB of two matrices A and B is well-defined if and only if the number of columns in A is equal to the number of lines in B. Therefore, the products CA, AE, BE and CD are not defined.

The general formula for the product of a matrix $A = (a_{ij})_{1\le i\le p, 1\le j\le n}$ with a matrix $B = (b_{kl})_{1\le k\le n, 1\le l\le m}$ is the following : $C = AB$ is a matrix of size (p, m) whose coefficients are : $c_{il} = \sum_{k=1}^{n} a_{ik} b_{kl}$ (the first index i is for the lines, $1 \le i \le p$, the second is for the columns, $1 \le l \le m$).

For example, with the matrices A and B given in the exercise, we have $p = 3$ (the number of lines in A), $n = 3$ (the numbers of columns in A and the number of lines in B), $m = 3$ (the numbers of columns in B), and the third coefficient on the first line of the product AB is

$$c_{13} = \sum_{k=1}^{3} a_{1k} b_{k3} = a_{11} b_{13} + a_{12} b_{23} + a_{13} b_{33} = 1 \times (-1) + 2 \times 4 + (-1) \times (-2) = 9.$$

One obtains the following results :

$$AB = \begin{pmatrix} 4 & 0 & 9 \\ 6 & 0 & 14 \\ 0 & 0 & 0 \end{pmatrix} \quad BA = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 1 & 2 & -1 \end{pmatrix} \quad AC = \begin{pmatrix} 1 & -1 \\ 1 & -3 \\ 0 & 0 \end{pmatrix} \quad AD = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} -1 & -3 \\ 16 & 0 \\ -4 & -4 \end{pmatrix} \quad BD = \begin{pmatrix} -1 \\ 16 \\ -4 \end{pmatrix} \quad DE = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{pmatrix}.$$

Moreover $ED = 7$.

Exercise 2 Consider the following matrices :

$$A = \begin{pmatrix} 2 & 5 & -1 \\ 0 & 1 & 3 \\ 0 & -2 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 7 & -1 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ -1 & 0 \end{pmatrix}.$$

Compute : $(A - 2B)C$, $C^T A$, $C^T B$, $C^T(A^T - 2B^T)$, where M^T denotes the transpose of M.

Solution of Exercise 2 : One has :

$$A - 2B = \begin{pmatrix} 0 & -9 & 1 \\ -4 & -5 & -5 \\ 0 & -2 & 4 \end{pmatrix} \quad (A-2B)C = \begin{pmatrix} -1 & -36 \\ 1 & -28 \\ -4 & -8 \end{pmatrix}.$$

$$C^T = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 4 & 0 \end{pmatrix} \quad C^T A = \begin{pmatrix} 2 & 7 & -5 \\ 4 & 14 & 10 \end{pmatrix} \quad C^T B = \begin{pmatrix} 1 & 7 & -1 \\ 10 & 26 & 14 \end{pmatrix}$$

$$A^T - 2B^T = (A-2B)^T = \begin{pmatrix} 0 & -4 & 0 \\ -9 & -5 & -2 \\ 1 & -5 & 4 \end{pmatrix} \quad C^T(A^T - 2B^T) = \begin{pmatrix} -1 & 1 & -4 \\ -36 & -28 & -8 \end{pmatrix}$$

As expected, we verify that $C^T(A^T - 2B^T) = \big((A-2B)C\big)^T$

Exercise 3 Compute A^n for all $n \in \mathbb{Z}$, with successively

$$A = \begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix}, \qquad \begin{pmatrix} \cosh(a) & \sinh(a) \\ \sinh(a) & \cosh(a) \end{pmatrix}.$$

Solution of Exercise 3 : Using the trigonometric formulas

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

(which we can recover by considering the real and imaginary parts of the equality $e^{i(a+b)} = e^{ia}e^{ib}$), we obtain by induction on n :

$$A^n = \begin{pmatrix} \cos(na) & -\sin(na) \\ \sin(na) & \cos(na) \end{pmatrix}.$$

Another way to obtain this result is the following : the matrix A is the matrix of the rotation of angle a in the plane. Composing A with itself n-times gives the rotation of angle na.

Consider now the second matrix. Recall that $\cosh(a) = \frac{e^a + e^a}{2}$ and $\sinh(a) = \frac{e^a - e^{-a}}{2}$; in other words, cosh is the even part of the exponential, and sinh is the odd part of the exponential. In particular, one has $\cosh^2(a) - \sinh^2(a) = 1$. Using the formulas

$$\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$$
$$\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)$$

(which we can recover by writing out everything in terms of the exponential function), we obtain, again by induction on n :

$$A^n = \begin{pmatrix} \cosh(na) & \sinh(na) \\ \sinh(na) & \cosh(na) \end{pmatrix}.$$

Exercise 4 Are the following matrices invertible ? If so, compute their inverses.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

Solution of Exercise 4 : Let us apply the Gauss algorithm to the matrix obtained by juxtaposing to the first matrix the identity matrix of the same size :

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \to L_2 - 2L_1 \\ L_3 \to L_3 - 3L_1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \rightarrow L_3 - 5L_2 \end{array}$$

On the left, we see a triangular matrix with nonzero diagonal coefficients. Hence the initial matrix is invertible. To compute its inverse, one uses the operations of the Gauss algorithm to obtain the identity matrix on the left. Once we have the identity matrix on the left, the inverse can be read on the right :

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \rightarrow -L_2 \\ L_3 \rightarrow \frac{1}{18}L_3 \end{array} \quad \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{3}{18} & \frac{15}{18} & -\frac{3}{18} \\ 0 & 1 & 0 & \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{array}\right) \quad \begin{array}{l} L_1 \rightarrow L_1 - 3L_3 \\ L_2 \rightarrow L_2 - 5L_3 \\ L_3 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ 0 & 1 & 0 & \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{array}\right) \quad \begin{array}{l} L_1 \rightarrow L_1 - 2L_2 \\ L_2 \\ L_3 \end{array}$$

Hence the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{pmatrix}$.

In the same manner, one obtains that the matrix $B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ is invertible, with inverse

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{5}{3} & \frac{-4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 5 & -4 & 3 \\ -2 & 1 & 0 \end{pmatrix}.$$

The matrix $C = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ is also invertible, with inverse $C^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}$

Exercise 5 Compute the inverses of the following matrices :

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & a & a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution of Exercise 5 : The inverse of $D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$, is

$$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The inverse of $E = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & a & a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$, is $E^{-1} = \begin{pmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

The inverse of $F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, is $F^{-1} = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Exercise 6 The *exponential* of a square matrix M is, by definition, the limit of the following sequence

$$e^M = 1 + M + \frac{M^2}{2!} + \cdots = \lim_{N\to+\infty} \sum_{k=0}^{N} \frac{M^k}{k!}.$$

One admits that this limit exists, by a theorem of Analysis.

1. Show that if $AB = BA$ then $e^{A+B} = e^A e^B$. One is allowed, to treat this question, to pass to the limit without justification.
2. Compute e^M for the four following matrices :

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

3. Find a simple example where $e^{A+B} \neq e^A e^B$.

Solution of Exercise 6 :

1. Under the hypothesis that $AB = BA$, one has :

$$(A+B)^n = \sum_{k=0}^{n} C_n^k A^k B^{n-k},$$

where $C_n^k = \frac{n!}{(n-k)!k!}$. Therefore

$$\begin{aligned} e^{A+B} &= \lim_{N\to+\infty} \textstyle\sum_{n=0}^{N} \frac{(A+B)^n}{n!} = \lim_{N\to+\infty} \sum_{n=0}^{N} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!\ k!} A^k B^{n-k} \\ &= \lim_{N\to+\infty} \textstyle\sum_{n=0}^{N} \sum_{k=0}^{n} \frac{A^k}{k!} \frac{B^{(n-k)}}{(n-k)!}. \end{aligned}$$

On the other hand,

$$e^A e^B = \left(\lim_{N_1\to+\infty} \sum_{k=0}^{N_1} \frac{A^k}{k!} \right) \cdot \left(\lim_{N_2\to+\infty} \sum_{l=0}^{N_2} \frac{B^l}{l!} \right).$$

Under the hypothesis that these two limits exists, one has :

$$e^A e^B = \lim_{N_1\to+\infty} \lim_{N_2\to+\infty} \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} \frac{A^k B^l}{k!\ l!}$$

Note that in this last expression the indices (k, l) take all values of $\mathbb{N} \times \mathbb{N}$. Another way to count all integers from $\mathbb{N} \times \mathbb{N}$ is to count the integers on the diagonal $k + l = n$ and make this diagonal vary from $n = 0$ to $n = +\infty$. This gives rise to the following equalities :

$$e^A e^B = \lim_{N\to+\infty} \sum_{n=0}^{N} \sum_{k=0}^{n} \frac{A^k}{k!} \frac{B^{(n-k)}}{(n-k)!} = e^{A+B}.$$

2. Let A be the first matrix. One has :

$$A^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix},$$

therefore

$$e^A = \begin{pmatrix} \lim_{N\to+\infty} \sum_{n=0}^{N} \frac{a^n}{n!} & 0 & 0 \\ 0 & \lim_{N\to+\infty} \sum_{n=0}^{N} \frac{b^n}{n!} & 0 \\ 0 & 0 & \lim_{N\to+\infty} \sum_{n=0}^{N} \frac{c^n}{n!} \end{pmatrix} = \begin{pmatrix} e^a & 0 & 0 \\ 0 & e^b & 0 \\ 0 & 0 & e^c \end{pmatrix}.$$

Let us denote by B the second matrix. One has :

$$B^2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $B^n = 0$ for $n \geq 3$. Therefore

$$e^B = 1 + B + \frac{B^2}{2!} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & \frac{ac}{2}+b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote by C the third matrix. It is the matrix of the rotation centered at the origin of angle $-\frac{\pi}{2}$. Since C^n is the matrix of the rotation centered at the origin of angle $n \times \frac{\pi}{2}$, one has :

$$C^{4k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C^{4k+1} = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C^{4k+2} = C^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C^{4k+3} = C^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore

$$\begin{array}{cccccccc} e^C & = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & + & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & + & \frac{1}{2!}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & + & \frac{1}{3!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ & + & \frac{1}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & + & \frac{1}{5!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & + & \frac{1}{6!}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & + & \frac{1}{7!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ & + & \cdots & + & \cdots & + & \cdots & + & \cdots \\ & + & \frac{1}{(4k)!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & + & \frac{1}{(4k+1)!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & + & \frac{1}{(4k+2)!}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & + & \frac{1}{(4k+3)!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ & + & \cdots & + & \cdots & + & \cdots & + & \cdots \end{array}$$

$$e^C = \begin{pmatrix} \lim_{N\to+\infty}\sum_{l=0}^N \frac{(-1)^l}{(2l)!} & \lim_{N\to+\infty}\sum_{l=0}^N \frac{(-1)^l}{(2l+1)!} \\ -\lim_{N\to+\infty}\sum_{l=0}^N \frac{(-1)^l}{(2l+1)!} & \lim_{N\to+\infty}\sum_{l=0}^N \frac{(-1)^l}{(2l)!} \end{pmatrix} = \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix},$$

where we have used the series :

$$\cos x = \lim_{N\to+\infty}\sum_{l=0}^N \frac{(-1)^l x^{2l}}{(2l)!}$$

$$\sin x = \lim_{N\to+\infty}\sum_{l=0}^N \frac{(-1)^l x^{2l+1}}{(2l+1)!}.$$

Denoting by D the fourth matrix, $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, one has :

$$e^D = \begin{pmatrix} e^1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Consider the matrix $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. One has $ED = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq DE = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Moreover $D + E = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and

$$(D+E)^n = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned} e^{D+E} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \tfrac{1}{2!}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \cdots + \tfrac{1}{n!}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} e^1 & e^1 - 1 \\ 0 & 1 \end{pmatrix} \end{aligned}.$$

On the other hand,

$$e^E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $e^D e^E = \begin{pmatrix} e^1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^1 & e^1 \\ 0 & 1 \end{pmatrix}$.