University of Lille I
PC first year **list of exercises n° 5**

Determinants

Exercise 1 1. Compute the area of the parallelogram constructed on the vectors $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

2. Compute the volume of the parallelepiped constructed on the following vectors $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
3. Show that the volume of a parallelepiped whose vertices are points in $\mathbb{R}^3$ with integer coordinates is an integer.

Solution of Exercise 1 :

1. The area of a parallelogram constructed on two vectors is the absolute value of the determinant of these two vectors. Hence the area of the parallelogram constructed on $\vec{a}$ and $\vec{b}$ is

$$\left|\det\,(\vec{a}, \vec{b})\right| = -\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = 3.$$

2. The volume of a parallelepiped constructed on three vectors is the absolute value of the determinant of these three vectors. Hence the volume of the parallelepiped constructed on $\vec{u}$, $\vec{v}$, $\vec{w}$ is

$$|\det\,(\vec{u}, \vec{v}, \vec{w})| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

3. The determinant is computed using only products and sums of numbers. Therefore the determinant of vectors with only integer coordinates is an integer. Consequently the same is true for the volume.

Exercise 2 Compute the determinants of the following matrices :

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 3 \\ 1 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Solution of Exercise 3 :

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 1 \end{vmatrix} = -12, \qquad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 2 & 1 & 2 \\ 3 & 1 & 3 \\ 1 & 0 & 6 \end{vmatrix} = -5, \qquad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -18, \qquad \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{vmatrix} = -3.$$

Exercise 3 Compute the determinants of the following matrices :

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ 2 & 1 & 0 & 6 \\ 1 & 1 & 1 & 7 \end{pmatrix}.$$

Solution of Exercise 3 :

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix} = 96, \qquad \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -1, \qquad \begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ 2 & 1 & 0 & 6 \\ 1 & 1 & 1 & 7 \end{vmatrix} = -12.$$

Exercise 4 Compute the determinants of the following matrices :

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 35 \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ -a & b & \alpha & \beta \\ -a & -b & c & \gamma \\ -a & -b & -c & d \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

Solution of Exercise 4 :

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 35 \end{vmatrix} = -3, \qquad \begin{vmatrix} a & b & c & d \\ -a & b & \alpha & \beta \\ -a & -b & c & \gamma \\ -a & -b & -c & d \end{vmatrix} = 8abc, \qquad \begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix} = -16.$$

Exercise 5 Compute the determinant of the following matrices :

$$\begin{pmatrix} a & b & c & d \\ e & f & g & 0 \\ h & k & 0 & 0 \\ l & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & k & l \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & k \\ 0 & 0 & 0 & l \end{pmatrix}.$$

Solution of Exercise 5 :

$$\begin{vmatrix} a & b & c & d \\ e & f & g & 0 \\ h & k & 0 & 0 \\ l & 0 & 0 & 0 \end{vmatrix} = lkgd, \qquad \begin{vmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & k & l \end{vmatrix} = acfl, \qquad \begin{vmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & k \\ 0 & 0 & 0 & l \end{vmatrix} = aehl.$$

Exercise 6 Let $M = (m_i^j)$ be a square matrix of size n. One constructs from M a matrix $N = (n_i^j)$ in the following way : for each pair of indices i, j, denote by M_i^j the matrix obtained from M by deleting the line i and the column j ; then $n_i^j = (-1)^{i+j}\det(M_i^j)$. Prove that $MN = NM = \det(M)I$, where I denotes the identity matrix. Using this result, find a way to invert a matrix that uses determinants and apply it to the following matrix :

$$M = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

Solution of Exercise 6 : Done in the lecture notes.

Exercise 7 Compute the inverses of the following matrices using two different methods :

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ -1 & 0 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Solution of Exercise 7 : One can use either the formula with the comatrix, or the Gauss method. One obtains :

$$A^{-1} = \begin{pmatrix} 4 & -2 & -1 \\ -\frac{9}{2} & \frac{5}{2} & \frac{1}{2} \\ 2 & -1 & 0 \end{pmatrix}; \qquad B^{-1} = \begin{pmatrix} -3 & 9 & -5 \\ 3 & -7 & 4 \\ -1 & 2 & -1 \end{pmatrix};$$

$$C^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & 0 & -\frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix}; \qquad D^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = D.$$

Exercise 8 Using the determinant, show that the following systems admit a unique solution. Solve each system by inverting the matrix of coefficients.

$$\begin{cases} x + y + z = 1 \\ 2x + 3y + 4z = 2 \\ y + 4z = 3 \end{cases}$$

$$\begin{cases} x + y + z + t = 1 \\ x + y - z - t = 0 \\ x - y - z + t = 2 \\ x - y + z - t = 3 \end{cases}$$

$$\begin{cases} 3x + 2y - z = -1 \\ x + y - 3z = 1 \\ 3x + 2y = 0 \end{cases}$$

Solution of Exercise 8 :

1. One has
$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 1 & 4 \end{pmatrix} = 2 \neq 0.$$
It follows that the matrix of coefficients of the system is invertible, and the system admits a unique solution. One computes
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -3 \\ \frac{3}{2} \end{pmatrix}.$$
2. One has
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = 16 \neq 0.$$
The unique solution of the second system is
$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$
3. One has
$$\begin{vmatrix} 3 & 2 & -1 \\ 1 & 1 & -3 \\ 3 & 2 & 0 \end{vmatrix} = 1 \neq 0.$$
The unique solution of the third system is
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 \\ 12 \\ 1 \end{pmatrix}.$$