University of Lille I
PC first year **list of exercises n° 6**

Eigenvalues and eigenvectors

Exercise 1 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3x+4y \\ 4x-3y \end{pmatrix}.$$

(a) Write the matrix A of f in the canonical basis of $\mathbb{R}^2$.

(b) Show that the vector $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of f. What is the associated eigenvalue ?

(c) Show that the vector $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is also an eigenvector of f. What is the associated eigenvalue ?

(d) Using a picture, determine the image of the vector $\vec{v}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Recover this result by a computation.

(e) Show that the family $\{\vec{v}_1, \vec{v}_2\}$ forms a basis of $\mathbb{R}^2$.

(f) What is the matrix of f in the basis $\{\vec{v}_1, \vec{v}_2\}$? Denote it by D.

(g) Let P be the matrix whose first column is the vector $\vec{v}_1$ and whose second column is the vector $\vec{v}_2$, expressed in the canonical basis of $\mathbb{R}^2$. Compute P^{-1}.

(h) What is the relationship between A, P, P^{-1} and D ?

(i) Compute A^n, for $n \in \mathbb{N}$.

2. Do the same exercise with $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ and the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Solution of Exercise 1 :

1. (a) The matrix of $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3x+4y \\ 4x-3y \end{pmatrix}$ in the canonical basis of $\mathbb{R}^2$ is

$$A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}.$$

(b) Apply f to $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. One has

$$f(\vec{v_1}) = \frac{1}{5}\begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \vec{v}_1.$$

Consequently $\vec{v_1}$ is an eigenvector of f associated to the eigenvalue $\lambda_1 = 1$.

(c) One has

$$f(\vec{v}_2) = A\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -\vec{v}_2.$$

Hence $\vec{v}_2$ is an eigenvector of f with eigenvalue $\lambda_2 = -1$.

(d) One has $\vec{v_3} = \vec{v_1} + \vec{v_2}$. Therefore the image of $\vec{v_3}$ by the linear map f is the sum of the images of $\vec{v_1}$ and $\vec{v_2}$:

$$f(\vec{v_3}) = f(\vec{v_1}) + f(\vec{v_2}) = \lambda_1\vec{v_1} + \lambda_2\vec{v_2} = \vec{v_1} - \vec{v_2}.$$

One obtains $f(\vec{v_3}) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(e) One has

$$\det\,(\vec{v_1}, \vec{v_2}) = \det\,\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = 5.$$

Since $\det\,(\vec{v_1}, \vec{v_2}) \neq 0$, the family $\{\vec{v_1}, \vec{v_2}\}$ forms a basis of $\mathbb{R}^2$.

(f) The matrix of f in the basis $\{\vec{v_1}, \vec{v_2}\}$ is

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that f is the reflection (or axial symmetry) with respect to $\mathbb{R}\vec{v_1}$ parallel to $\mathbb{R}\vec{v_2}$. Moreover one can check that $\vec{v_1} \perp \vec{v_2}$, hence f is an *orthogonal* reflection.

(g) The inverse of the matrix

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

is the matrix

$$P^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

(h) The relationship between A, P, P^{-1} and D is $D = P^{-1}AP$.

(i) One has $A^n = I$, for n even, and $A^n = A$ for n odd (here I denotes the identity matrix). This is coherent with the fact that f is a reflection.

2. (a) Let us apply A to $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. One has

$$A\vec{v_1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\vec{v_1}.$$

Consequently $\vec{v_1}$ is an eigenvector of A associated to the eigenvalue $\lambda_1 = 2$.

(b) One has

$$A\vec{v_2} = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{v_2}.$$

Hence $\vec{v_2}$ is an eigenvector of A with eigenvalue $\lambda_2 = 1$.

(c) One has $\vec{v_3} = -\vec{v_1} + \vec{v_2}$. Therefore

$$f(\vec{v_3}) = -f(\vec{v_1}) + f(\vec{v_2}) = -\lambda_1\vec{v_1} + \lambda_2\vec{v_2} = -2\vec{v_1} + \vec{v_2}.$$

One obtains $f(\vec{v_3}) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$.

(d) One has $\det\,(\vec{v_1}, \vec{v_2}) = \det\,\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1$. Since $\det\,(\vec{v_1}, \vec{v_2}) \neq 0$, the family $\{\vec{v_1}, \vec{v_2}\}$ forms a basis of $\mathbb{R}^2$.

(e) The matrix of the linear map f associated to A, written in the basis $\{\vec{v_1}, \vec{v_2}\}$, is

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(f) The inverse of the matrix

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is the matrix

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

(g) The relationship between A, P, P^{-1} and D is $D = P^{-1}AP$, which is equivalent to $A = PDP^{-1}$.

(h) One has

$$D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, for $n \in \mathbb{N}$, one has $A^n = PD^nP^{-1}$. Hence for $n \in \mathbb{N}$,

$$A^n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$

Exercise 2 Determine the characteristic polynomial of the following matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Solution of Exercise 2 :

1. For the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

one has

$$\det\,(A - \lambda I) = \lambda^2 - 1.$$

2. For the matrix

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

one has

$$\det\,(B - \lambda I) = -\lambda^3 + 3\lambda + 2.$$

3. For the matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

one has

$$\det\,(C - \lambda I) = \lambda^4 - 6\lambda^2 - 8\lambda - 3.$$

Exercise 3 Find the eigenvalues and a basis of eigenvectors of the following matrices :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 4 \\ 0 & 7 & -2 \\ 4 & -2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & -1 \\ -1 & a^2 & 0 \\ -1 & 0 & a^2 \end{pmatrix} \quad (a \neq 0).$$

Solution of Exercise 3 :

1. Consider the matrix
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

   One has
$$\det\,(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)(-1-\lambda)-1\right) = (1-\lambda)(\lambda-\sqrt{2})(\lambda+\sqrt{2}).$$

   It follows that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = \sqrt{2}$ and $\lambda_3 = -\sqrt{2}$. An eigenvector associated to the eigenvalue $\lambda_1 = 1$ is a nonzero element of the kernel of $A - \lambda_1 I$. One has
$$A - \lambda_1 I = A - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

   Therefore the vector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of A associated to the eigenvalue $\lambda_1 = 1$.

   For $\lambda_2 = \sqrt{2}$, one has
$$A - \lambda_2 I = A - \sqrt{2}I = \begin{pmatrix} 1-\sqrt{2} & 0 & 0 \\ 0 & 1-\sqrt{2} & 1 \\ 0 & 1 & -1-\sqrt{2} \end{pmatrix}.$$

   Denoting by C_1, C_2, C_3 the columns of the above matrix, one sees that $-C_2 + (1-\sqrt{2})C_3$ is the null–vector. Therefore the vector $\vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1-\sqrt{2} \end{pmatrix}$ is an eigenvector of A associated to the eigenvalue $\lambda_2 = \sqrt{2}$.

   For $\lambda_3 = -\sqrt{2}$, one has
$$A - \lambda_3 I = A + \sqrt{2}I = \begin{pmatrix} 1+\sqrt{2} & 0 & 0 \\ 0 & 1+\sqrt{2} & 1 \\ 0 & 1 & -1+\sqrt{2} \end{pmatrix}.$$

   Denoting by C_1, C_2, C_3 the columns of the above matrix, one sees that $-C_2+(1+\sqrt{2})C_3$ is the null–vector. Therefore the vector $\vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1+\sqrt{2} \end{pmatrix}$ is an eigenvector of A associated to the eigenvalue $\lambda_3 = -\sqrt{2}$.

2. Consider the matrix
$$B = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 7 & -2 \\ 4 & -2 & 0 \end{pmatrix}.$$

   Let us compute
$$\det\,(B-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 7-\lambda & -2 \\ 4 & -2 & -\lambda \end{vmatrix} = (1-\lambda)(7-\lambda)(-\lambda) - 16(7-\lambda) + 4(1-\lambda)$$
$$= -\lambda^3 - 8\lambda^2 + 13\lambda - 116 = (\lambda-4)(-\lambda^2+4\lambda+29) = -(\lambda-4)(\lambda-2-\sqrt{33})(\lambda-2+\sqrt{33}).$$

It follows that the eigenvalues of B are $\lambda_1 = 4$, $\lambda_2 = 2 - \sqrt{33}$ and $\lambda_3 = 2 + \sqrt{33}$.

For $\lambda_1 = 4$, one has

$$\begin{pmatrix} A - \lambda_1 I \\ \\ I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -3 & 0 & 4 \\ 0 & 3 & -2 \\ 4 & -2 & -4 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \xrightarrow{C_3 \leftarrow 3C_3 + 4C_1} \begin{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & -6 \\ 4 & -2 & 4 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{pmatrix} \xrightarrow{C_3 \leftarrow C_3 + 2C_2} \begin{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \end{pmatrix}$$

It follows that the vector $\vec{v}_1 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$ is an eigenvector of B associated to the eigenvalue $\lambda_1 = 4$.

For $\lambda_2 = 2 - \sqrt{33}$, one has

$$\begin{pmatrix} A - \lambda_2 I \\ \\ I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -1+\sqrt{33} & 0 & 4 \\ 0 & 5+\sqrt{33} & -2 \\ 4 & -2 & -2+\sqrt{33} \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \xrightarrow{C_3 \leftarrow (-1+\sqrt{33})C_3 - 4C_1} \begin{pmatrix} -1+\sqrt{33} & 0 & 0 \\ 0 & 5+\sqrt{33} & 2-2\sqrt{33} \\ 4 & -2 & 19-3\sqrt{33} \\ \\ 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & (-1+\sqrt{33}) \end{pmatrix}$$

$$\xrightarrow{C_3 \leftarrow (5+\sqrt{33})C_3 - (2-2\sqrt{33})C_2} \begin{pmatrix} -1+\sqrt{33} & 0 & 0 \\ 0 & 5+\sqrt{33} & 0 \\ 4 & -2 & 0 \\ \\ 1 & 0 & -20-4\sqrt{33} \\ 0 & 1 & -2+2\sqrt{33} \\ 0 & 0 & 28+4\sqrt{33} \end{pmatrix}$$

Consequently, the vector $\vec{v}_2 = \begin{pmatrix} -20-4\sqrt{33} \\ -2+2\sqrt{33} \\ 28+4\sqrt{33} \end{pmatrix}$ generates the eigenspace associated to the eigenvalue $\lambda_2 = 2 - \sqrt{33}$.

The same computations with $\sqrt{33}$ instead of $-\sqrt{33}$ show that the vector $\vec{v}_3 = \begin{pmatrix} -20+4\sqrt{33} \\ -2-2\sqrt{33} \\ 28-4\sqrt{33} \end{pmatrix}$ generates the eigenspace associated to the eigenvalue $\lambda_3 = 2 + \sqrt{33}$.

3. Consider the matrix

$$C = \begin{pmatrix} 1 & -1 & -1 \\ -1 & a^2 & 0 \\ -1 & 0 & a^2 \end{pmatrix} \quad (a \neq 0).$$

One has

$$\det\,(C - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & a^2-\lambda & 0 \\ -1 & 0 & a^2-\lambda \end{vmatrix} = (a^2 - \lambda)\left(\lambda^2 - (a^1+1)\lambda + a^2 - 2\right)$$

$$= (a^2 - \lambda)\left(\lambda - \frac{a^2+1+\sqrt{a^4-2a^2+9}}{2}\right)\left(\lambda - \frac{a^2+1-\sqrt{a^4-2a^2+9}}{2}\right).$$

Therefore the eigenvalues of C are $\lambda_1 = a^2$, $\lambda_2 = \frac{a^2+1+\sqrt{a^4-2a^2+9}}{2}$ and $\lambda_3 = \frac{a^2+1-\sqrt{a^4-2a^2+9}}{2}$.

An eigenvector associated to the eigenvalues $\lambda_1 = a^2$ is $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

For $\lambda_2 = \frac{a^2+1+\sqrt{a^4-2a^2+9}}{2}$, one has

$$\begin{pmatrix} A - \lambda_2 I \\ \\ I \end{pmatrix} = \begin{pmatrix} \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} & -1 & -1 \\ -1 & \frac{a^2-1-\sqrt{a^4-2a^2+9}}{2} & 0 \\ -1 & 0 & \frac{a^2-1-\sqrt{a^4-2a^2+9}}{2} \\ & & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} C_2 \leftarrow \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} C_2 + C_1 \\ C_3 \leftarrow \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} C_3 + C_1 \end{matrix}$$

$$\to \begin{pmatrix} \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ & & \\ 1 & 1 & 1 \\ 0 & \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} & 0 \\ 0 & 0 & \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} \end{pmatrix}$$

$$C_3 \leftarrow C_3 + C_2$$

$$\to \begin{pmatrix} \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ & & \\ 1 & 1 & 2 \\ 0 & \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} & \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} \\ 0 & 0 & \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} \end{pmatrix}$$

It follows that an eigenvector associated to the eigenvalue $\lambda_2 = \frac{a^2+1+\sqrt{a^4-2a^2+9}}{2}$ is $\vec{v}_2 = \begin{pmatrix} 2 \\ \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} \\ \frac{1-a^2-\sqrt{a^4-2a^2+9}}{2} \end{pmatrix}$.

For $\lambda_3 = \frac{a^2+1-\sqrt{a^4-2a^2+9}}{2}$, one finds similarly that an eigenvector is given by $\vec{v}_3 = \begin{pmatrix} 2 \\ \frac{1-a^2+\sqrt{a^4-2a^2+9}}{2} \\ \frac{1-a^2+\sqrt{a^4-2a^2+9}}{2} \end{pmatrix}$.

Exercise 4 Find an invertible matrix P such that $B = PAP^{-1}$ is diagonal, and write the corresponding matrix B , for the following matrices A :

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & -4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 4 \\ 0 & 7 & -2 \\ 4 & -2 & 0 \end{pmatrix}.$$

Solution of Exercise 4

1. Consider the first matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & -4 & 1 \end{pmatrix}.$$

One has

$$\det\,(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -4 \\ 0 & -4 & 1-\lambda \end{vmatrix} = (2-\lambda)\left((1-\lambda)^2 - 16\right)$$

$$= (2-\lambda)\left(\lambda^2 - 2\lambda - 15\right) = (2-\lambda)(\lambda-5)(\lambda+3).$$

It follows that the eigenvalues of the first matrix are $\lambda_1 = 2$, $\lambda_2 = 5$ and $\lambda_3 = -3$. An eigenvector associated to $\lambda_1 = 2$ is $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Moreover one has

$$A - 5I = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{pmatrix},$$

hence an eigenvector associated to $\lambda_2 = 5$ is $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Similarly, since

$$A + 3I = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{pmatrix},$$

an eigenvector associated to $\lambda_3 = -3$ is $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. The matrix of coordinate change from the canonical basis of $\mathbb{R}^3$ to the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Using the change of coordinates formula, one obtains $B = P^{-1}AP$ with $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

2. Consider the second matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

One has

$$\det\,(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - (1-\lambda) = \lambda(1-\lambda)(\lambda-2).$$

Hence the eigenvalues of the second matrix are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 2$. A non-trivial vector in the kernel of A is $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. An eigenvector associated to $\lambda_2 = 1$ is $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Finally, an eigenvector associated to $\lambda_3 = 2$ is $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The matrix of coordinate change from the canonical basis of $\mathbb{R}^3$ to the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Using the change of coordinates formula, one obtains $B = P^{-1}AP$ with $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

3. Consider the third matrix

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 7 & -2 \\ 4 & -2 & 0 \end{pmatrix}.$$

Using Exercise 3, one has $A = PBP^{-1}$ with

$$P = \begin{pmatrix} 4 & -20-4\sqrt{33} & -20+4\sqrt{33} \\ 2 & -2+2\sqrt{33} & -2-2\sqrt{33} \\ 3 & 28+4\sqrt{33} & 28-4\sqrt{33} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2-\sqrt{33} & 0 \\ 0 & 0 & 2+\sqrt{33} \end{pmatrix}.$$

Exercise 5 (DS mai 2008) Consider the matrix

$$A = \begin{pmatrix} 7 & 3 & -9 \\ -2 & -1 & 2 \\ 2 & -1 & -4 \end{pmatrix}$$

which represents an endomorphism f of $\mathbb{R}^3$ expressed in the canonical basis $\mathcal{B} = \{\vec{i}, \vec{j}, \vec{k}\}$.

1. (a) Show that the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 3$.
   (b) Why can we diagonalize A ?
2. (a) Determine a basis $\mathcal{B}' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of eigenvectors such that the matrix of f in the basis $\mathcal{B}'$ is

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

   (b) Give the matrix P of coordinate change from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$; what is the relation between A, P, P^{-1} and D ?
3. Show that for any integer $n \in \mathbb{N}$, one has $A^n = PD^nP^{-1}$.
4. After having given D^n, compute A^n for all $n \in \mathbb{N}$.

Solution of Exercise 5

1. (a) Let us compute the eigenvalues of A. One has

$$\det\,(A - \lambda I) = \begin{vmatrix} 7-\lambda & 3 & -9 \\ -2 & -1-\lambda & 2 \\ 2 & -1 & -4-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = (\lambda+2)(1-\lambda)(\lambda-3).$$

It follows that the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 3$.

(b) Since the eigenvalues of A are real and distinct, A is diagonalizable over $\mathbb{R}$.

2. (a) One has

$$A + 2I = \begin{pmatrix} 9 & 3 & -9 \\ -2 & 1 & 2 \\ 2 & -1 & -2 \end{pmatrix}$$

It follows that the vector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_1 = -2$.

For $\lambda_2 = 1$, one has

$$A - I = \begin{pmatrix} 6 & 3 & -9 \\ -2 & -2 & 2 \\ 2 & -1 & -5 \end{pmatrix}$$

One finds that the eigenspace associated to $\lambda_2 = 1$ is generated by $\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Finally, for $\lambda_3 = 3$, one obtains

$$A - 3I = \begin{pmatrix} 4 & 3 & -9 \\ -2 & -4 & 2 \\ 2 & -1 & -7 \end{pmatrix},$$

and one finds $\vec{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ as eigenvector associated to $\lambda_3 = 3$. Consequently, the basis $\mathcal{B}' = \left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis of eigenvectors and the matrix of f in this basis is

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) The matrix P of coordinate change from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$ is

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The relationship between A, P, P^{-1} and D is $D = P^{-1}AP$.

3. We will show by induction that for any integer $n \in \mathbb{N}^*$, one has $A^n = PD^nP^{-1}$.
   - We will initialize the induction with $n = 1$. By the previous question, one has $D = P^{-1}AP$. Multiplying this identity by P^{-1} to the left and by P to the right, it follows that $PDP^{-1} = PP^{-1}APP^{-1} = A$. Hence the relation is satisfied for $n = 0$.
   - Suppose that, for a given rank k, the relation $A^k = PD^kP^{-1}$ is satisfied. Let us show that the relation is satisfied for the rank $k + 1$. One has

$$A^{k+1} = A^k A = PD^kP^{-1}A$$

by the induction hypothesis. Moreover, by the first step, one has $A = PDP^{-1}$. It follows that

$$A^{k+1} = PD^kP^{-1}PDP^{-1}.$$

Using $P^{-1}P = I$ and $ID = D$, one has

$$A^{k+1} = PD^kDP^{-1} = PD^{k+1}P^{-1}.$$

Consequently, the identity $A^k = PD^kP^{-1}$ for a given k implies the identity $A^{k+1} = PD^{k+1}P^{-1}$ for the rank $k+1$.

– By induction, the relationship $A^n = PD^nP^{-1}$ is satisfied for any integer $n \geq 1$.

4. One has
$$D^n = \begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{pmatrix}.$$
The inverse matrix of P is
$$P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$
It follows that
$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2+3^{n+1} & (-2)^n - 4 + 3^{n+1} & (-2)^n + 2 - 3^{n+1} \\ 1-3^n & 2-3^n & -1+3^n \\ -1+3^n & (-2)^n - 2 + 3^n & (-2)^n + 1 - 3^n \end{pmatrix}. \end{aligned}$$

Exercise 6 (DS mai 2008) Consider the matrix $A = \begin{pmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.

1. Compute the eigenvalues of A.
2. (a) Give a basis and the dimension of each eigenspace of A.
   (b) A is diagonalizable ; justify this claim and diagonalize A.

Solution of Exercise 6

1. Let us compute the eigenvalues of A. One has
$$\det\,(A - \lambda I) = \begin{vmatrix} -3-\lambda & -2 & -2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)^2.$$
It follows that $\lambda_1 = 1$ is a simple eigenvalue, and $\lambda_2 = -1$ is an eigenvalue with multiplicity 2.

2. (a) Consider
$$A + I = \begin{pmatrix} -2 & -2 & -2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$
The vectors $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ form a basis of the eigenspace associated to $\lambda_2 = -1$. The dimension of this eigenspace is therefore 2.

Consider
$$A - I = \begin{pmatrix} -4 & -2 & -2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$
One sees that the vector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to $\lambda_1 = 1$. The dimension of the eigenspace associated to $\lambda = 1$ is 1.

(b) Since the vectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^3$ (one can check for instance that $\det(\vec{v}_1, \vec{v}_2, \vec{v}_3) \neq 0$), A is diagonalizable. In the basis $\mathcal{B}' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, the matrix of the linear map whose matrix in the canonical basis of $\mathbb{R}^3$ is A is the following diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Exercise 7 Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a' & b & 2 & 0 \\ a'' & b' & c & 2 \end{pmatrix}.$$

Under which conditions on the unknowns is the matrix A diagonalizable ? These conditions being satisfied, give a basis of eigenvectors for A.

Solution of Exercise 7 : The eigenvalues of A are $\lambda_1 = 1$ with multiplicity 2, and $\lambda_2 = 2$ with multiplicity 2. One has

$$A - I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ a' & b & 1 & 0 \\ a'' & b' & c & 1 \end{pmatrix}$$

The kernel of $A - I$ is of dimension 2 if and only if $a = 0$. On the other hand, consider

$$A - 2I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ a' & b & 0 & 0 \\ a'' & b' & c & 0 \end{pmatrix}$$

The kernel of $A - 2I$ is of dimension 2 if and only if $c = 0$. Consequently, A is diagonalizable if and only if $a = c = 0$. The vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -a' \\ -a'' \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -b \\ -b' \end{pmatrix}$ form a basis of the eigenspace associated to the eigenvalue $\lambda_2 = 1$. The vectors $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of the eigenspace associated to the eigenvalue $\lambda_2 = 2$.