University of Lille I
PC first year **list of exercises n° 7**

Review

Exercise 1 1. Solve the following systems in 4 different ways (by substitution, by the Gauss method, by inverting the matrix of coefficients of the system, by Cramer's formulas) :

$$\begin{cases} 2x + y = 1 \\ 3x + 7y = 0 \end{cases}$$

2. Choose the method that seems the quickest to you and solve, according to the values of a, the following systems :

$$\begin{cases} ax + y = 2 \\ (a^2+1)x + 2ay = 1 \end{cases}$$

$$\begin{cases} (a+1)x + (a-1)y = 1 \\ (a-1)x + (a+1)y = 1 \end{cases}$$

Solution of Exercise 1 :

1. (a) By substitution

$$\begin{cases} 2x + y = 1 \\ 3x + 7y = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + y = 1 \\ x = -\frac{7}{3}y \end{cases} \Leftrightarrow \begin{cases} -\frac{14}{3}y + y = 1 \\ x = -\frac{7}{3}y \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -\frac{3}{11} \\ x = \frac{7}{11} \end{cases}$$

(b) By the Gauss method

$$\begin{cases} 2x + y = 1 \\ 3x + 7y = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + y = 1 \\ 11y = -3 \quad L_2 \leftarrow 2L_2 - 3L_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{1-y}{2} \\ y = -\frac{3}{11} \end{cases} \Leftrightarrow \begin{cases} y = -\frac{3}{11} \\ x = \frac{7}{11} \end{cases}$$

(c) The inverse of the matrix of coefficients of the system is

$$\begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix}^{-1} = \frac{1}{11}\begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix}.$$

Hence the solution of the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ -\frac{3}{11} \end{pmatrix}$$

(d) By Cramer's formulas

$$x = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}} = \frac{7}{11} \qquad y = \frac{\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}} = -\frac{3}{11}$$

2. The determinant of the first system is

$$\begin{vmatrix} a & 1 \\ (a^2+1) & 2a \end{vmatrix} = a^2 - 1.$$

(a) If $a \notin \{1, -1\}$, one can use Cramer's formulas to obtain :

$$\left\{ \begin{array}{rcrcl} ax & + & y & = & 2 \\ (a^2+1)x & + & 2ay & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x = \dfrac{\begin{vmatrix} 2 & 1 \\ 1 & 2a \end{vmatrix}}{\begin{vmatrix} a & 1 \\ (a^2+1) & 2a \end{vmatrix}} = \dfrac{4a-1}{a^2-1} \\ \\ y = \dfrac{\begin{vmatrix} a & 2 \\ (a^2+1) & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ (a^2+1) & 2a \end{vmatrix}} = \dfrac{-2a^2+a-2}{a^2-1} \end{array} \right.$$

(b) If $a = 1$, the system becomes

$$\left\{ \begin{array}{rcrcl} x & + & y & = & 2 \\ 2x & + & 2y & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcrcrl} x & + & y & = & 2 & \\ & & 0 & = & -1 & L_2 \leftarrow L_2 - 2L_1 \end{array} \right.$$

which is impossible.

(c) If $a = -1$, the system becomes

$$\left\{ \begin{array}{rcrcl} -x & + & y & = & 2 \\ 2x & - & 2y & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcrcrl} x & + & y & = & 2 & \\ & & 0 & = & 5 & L_2 \leftarrow L_2 + 2L_1 \end{array} \right.$$

which is also impossible.

The determinant of the second system is

$$\begin{vmatrix} (a+1) & (a-1) \\ (a-1) & (a+1) \end{vmatrix} = 4a.$$

(a) If $a \neq 0$, one can use Cramer's formulas to obtain :

$$\left\{ \begin{array}{rcrcl} (a+1)x & + & (a-1)y & = & 1 \\ (a-1)x & + & (a+1)y & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x = \dfrac{\begin{vmatrix} 1 & (a-1) \\ 1 & (a+1) \end{vmatrix}}{4a} = \dfrac{1}{2a} \\ \\ y = \dfrac{\begin{vmatrix} (a+1) & 1 \\ (a-1) & 1 \end{vmatrix}}{4a} = \dfrac{1}{2a} \end{array} \right.$$

(b) If $a = 0$, the system becomes

$$\left\{ \begin{array}{rcrcl} x & - & y & = & 1 \\ -x & + & y & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcrcrl} x & - & y & = & 1 & \\ & & 0 & = & 2 & L_2 \leftarrow L_2 + L_1 \end{array} \right.$$

which is impossible.

Exercise 2 Solve the following system of 5 equations with 6 unknowns :

$$\left\{ \begin{array}{rcrcrcrcrcrl} 2x & + & y & + & z & - & 2u & + & 3v & - & w & = 1 \\ 3x & + & 2y & + & 2z & - & 3u & + & 5v & - & 3w & = 4 \\ 2x & + & 2y & + & 2z & - & 2u & + & 4v & - & 4w & = 6 \\ x & + & y & + & z & - & u & + & 2v & - & 2w & = 3 \\ 3x & & & & & - & 3u & + & 3v & + & 3w & = -6 \end{array} \right.$$

Solution of Exercise 2 : By the Gauss method

$$\left\{\begin{array}{ccccccccccccl} 2x & + & y & + & z & - & 2u & + & 3v & - & w & = 1 \\ 3x & + & 2y & + & 2z & - & 3u & + & 5v & - & 3w & = 4 \\ 2x & + & 2y & + & 2z & - & 2u & + & 4v & - & 4w & = 6 \\ x & + & y & + & z & - & u & + & 2v & - & 2w & = 3 \\ 3x & & & & & - & 3u & + & 3v & + & 3w & = -6 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{ccccccccccccl} x & + & y & + & z & - & u & + & 2v & - & 2w & = 3 & L_1 \leftrightarrow L_4 \\ 3x & + & 2y & + & 2z & - & 3u & + & 5v & - & 3w & = 4 & \\ 2x & + & 2y & + & 2z & - & 2u & + & 4v & - & 4w & = 6 & \\ 2x & + & y & + & z & - & 2u & + & 3v & - & w & = 1 & \\ 3x & & & & & - & 3u & + & 3v & + & 3w & = -6 & \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{ccccccccccccl} x & + & y & + & z & - & u & + & 2v & - & 2w & = 3 & \\ & - & y & - & z & & & - & v & + & 3w & = -5 & L_2 \leftarrow L_2 - 3L_1 \\ & & & & & & & & & & 0 & = 0 & L_3 \leftarrow L_3 - 2L_1 \\ & - & y & - & z & & & - & v & + & 3w & = -5 & L_4 \leftarrow L_4 - 2L_1 \\ & - & 3y & - & 3z & & & - & 3v & + & 9w & = -15 & L_5 \leftarrow L_5 - 3L_1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{ccccccccccccl} x & + & y & + & z & - & u & + & 2v & - & 2w & = 3 \\ & - & y & - & z & & & - & v & + & 3w & = -5 \end{array}\right.$$

It follows that the set of solutions is a 4-space in $\mathbb{R}^6$. Let us parametrize the set of solutions by $a = z \in \mathbb{R}$, $b = u \in \mathbb{R}$, $c = v \in \mathbb{R}$, $d = w \in \mathbb{R}$. One obtains

$$\left\{\begin{array}{ccl} x & = & -y - a + b - 2c + 2d + 3 = b - c - d - 2 \\ y & = & -a - c + 3d + 5 \\ z & = & a \\ u & = & b \\ v & = & c \\ w & = & d \end{array}\right.$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Exercise 3 For each pair (A_i, b_i), $1 \leq i \leq 5$ of matrices below

1. give the nature of the set of solutions of the system $A_i X = b_i$;
2. give a parametric representation of the set of solutions of $A_i X = b_i$;
3. give a basis of the range and a basis of the kernel of A_i.

a) $A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$; b) $A_2 = \begin{pmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ $b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$;

c) $A_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $b_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$; d) $A_4 = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $b_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$;

e) $A_5 = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $b_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$;

Solution of Exercise 3 :

a) Since $\det A_1 = 1 \neq 0$, the matrix A_1 is invertible hence defines an isomorphism of $\mathbb{R}^4$. The system $A_1X = b_1$ has therefore a unique solution given by $X = A_1^{-1}b_1 = (0,0,-1,1)^T$ by a standard computation. The range of A_1 is $\mathbb{R}^4$, hence the canonical basis of $\mathbb{R}^4$ is a basis of Im A_1. The kernel of A_1 is $\{\overrightarrow{0}\}$, hence a basis of $\ker A_1$ is $\emptyset$.

b) The rank of A_2 is 4, hence the dimension of the kernel of A_2 is 1. Therefore the set of solutions of $A_2X = b_2$ is an affine line in $\mathbb{R}^5$ parallel to $\ker A_2$. Denote by (x, y, z, t, u) the coordinates in $\mathbb{R}^5$. Let us parametrize the set of solutions by $a = u \in \mathbb{R}$. The system is equivalent to

$$\left\{\begin{array}{rcl} x + 2y + t &=& 1-3a \\ y + z + t &=& 1-2a \\ z + 2t &=& 1-3a \\ t &=& 1-a \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x &=& 1-3a-2y-t \\ y &=& 1-2a-z-t \\ z &=& 1-3a-2t \\ t &=& 1-a \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcl} x &=& 1-3a-2-1+a = -2-2a \\ y &=& 1-2a+1+a+-(1-a) = 1 \\ z &=& 1-3a-2+2a = -1-a \\ t &=& 1-a \end{array}\right. \Leftrightarrow \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, a \in \mathbb{R}.$$

Since A_2 is surjective, the canonical basis of $\mathbb{R}^4$ is a basis of $\mathrm{Im}A_2$. The previous resolution implies that a basis of $\ker A_2$ is given by the single vector $\begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.

c) Since the last equation of the system is impossible, the system $A_3X = b_3$ admits no solution. The rank of A_3 is 4, therefore by the Rank theorem, the dimension of $\ker A_3$ is 0. A basis of $\mathrm{Im}A_3$ is given by the 4 columns of A_3. A basis of $\ker A_3$ is given by the empty set $\emptyset$.

d) The last equation of $A_4X = b_4$ is impossible, hence this system admits no solution. The rank of A_4 is 4, hence by the Rank theorem, the dimension of the kernel of A_4 is 1. A basis of $\mathrm{Im}A_4$ is given by the first 4 columns of A_4. A basis of $\ker A_4$ is a nontrivial vector $X \in \mathbb{R}^5$ solution of $A_4X = \overrightarrow{0}$. One finds that $\begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ generates $\ker A_4$.

e) For the basis of $\text{Im}A_5$ and $\ker A_5$ see d). The vector b_5 belongs to $\text{Im}A_5$ since the last equation (compatibility condition) is satisfied. The kernel of A_5 being a line, the set of solutions of $A_5X = b_5$ is an affine line in $\mathbb{R}^5$ parallel to $\ker A_5$. Since the vector $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is a particular solution of the system, one obtains that the set of solutions is parametrized by

$$\begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, a \in \mathbb{R}.$$

Exercise 4 Compute a basis of the image and a basis of the kernel of the linear application

$$\begin{array}{cccc} f : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^5 \\ & (x,y,z) & \longmapsto & (x+y, x+y+z, 2x+y+z, 2x+2y+z, y+z) \end{array}$$

What is the rank of f ?

Solution of Exercise 4 : The matrix of the linear application f is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let us compute a basis of $\text{Im}f$ and a basis of $\ker f$. One has :

$$\begin{array}{c} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Consequently the kernel of f is trivial, and a basis of $\text{Im}f$ is given by $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. The rank of f is the dimension of $\text{Im}f$, that is, 3.

Exercise 5 Let A be the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

1. Consider the matrices $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}$. Show that $AB = AC$. Can the matrix A be invertible ?
2. Determine all matrices F of size $(3, 3)$ such that $AF = 0$ (where 0 denotes the matrix all of whose entries are zero).

Solution of Exercise 5 :

1. One has
$$AB = AC = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 4 & 3 \end{pmatrix}.$$
Suppose that the matrix A is invertible. Multiply both members of the equation $AB = AC$ on the left by A^{-1} to get $B = C$. But the matrices B and C are not equal. This is a contradiction. Hence the matrix A is not invertible.
2. Let F be any real matrix $(3, 3)$
$$F = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$
The equation $AF = 0$ gives rise to the following system
$$\begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d + g = 0 \\ e + h = 0 \\ f + i = 0 \\ 3a + d + g = 0 \\ 3b + e + h = 0 \\ 3c + f + i = 0 \end{cases}$$
Consequently the set of matrices F such that $AF = 0$ is the set of matrices of the form
$$F = \begin{pmatrix} 0 & 0 & 0 \\ d & e & f \\ -d & -e & -f \end{pmatrix}, d \in \mathbb{R}, e \in \mathbb{R}, f \in \mathbb{R}.$$

Exercise 6 For which values of a is the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{pmatrix}$$
invertible ? Compute in this case its inverse.

Solution of Exercise 6 : One has
$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & a \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & a \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 2a - 12 - (a - 3) + 2 = a - 7.$$
Hence A is invertible if and only if $a \neq 7$. In this case, the standard algorithm yields
$$A^{-1} = \frac{1}{a-7} \begin{pmatrix} 2a - 12 & 3 - a & 2 \\ 4 - a & a - 1 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Exercise 7 Let a and b be two real numbers, and A be the matrix

$$A = \begin{pmatrix} a & 2 & -1 & b \\ 3 & 0 & 1 & -4 \\ 5 & 4 & -1 & 2 \end{pmatrix}$$

Show that $\mathrm{rk}(A) \geq 2$ (where rk denotes the rank). For which values of a and b is the rank of A equal to 2 ?

Solution of Exercise 7 :
Recall that the rank of A is the greatest number of columns of A that are linearly independent. Since the second and third columns C_2, C_3 of A are not proportional, they are linearly independent. Therefore the rank of A is at least 2. For the rank of A to be exactly 2, one has to impose that the first and last columns of A are each a linear combination of C_2 and C_3 (which are fixed). The only linear combination of C_2 and C_3 that has the form $(a, 3, 5)^T$ is $3C_3 + 2C_2 = (1, 3, 5)^T$, hence $a = 1$. The only linear combination of C_2 and C_3 that has the form $(b, -4, 2)^T$ is $-4C_3 - \frac{1}{2}C_2 = (3, -4, 2)^T$, hence $b = 3$. Consequently the rank of A is 2 if and only if $a = 1$ and $b = 3$.

Exercise 8 Compute the inverse of the following matrix

$$A = \begin{pmatrix} 4 & 8 & 7 & 4 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Solution of Exercise 8 : One obtains

$$A^{-1} = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 4 & -4 \\ 1 & 0 & -4 & 5 \end{pmatrix}.$$

Exercise 9 Let us denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$. To a permutation $\sigma \in \mathcal{S}_n$, one associates the following endomorphism u_σ of $\mathbb{R}^n$:

$$\begin{array}{cccc} u_\sigma : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \longmapsto & \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix} \end{array}$$

1. Let $\tau = (ij)$ be a transposition. Write the matrix of u_τ in the canonical basis. Show that $\det(u_\tau) = -1$.
2. Show that $\forall \sigma, \sigma' \in \mathcal{S}_n$, $u_\sigma \circ u_{\sigma'} = u_{\sigma' \circ \sigma}$. **Caution ! There was a typo in the French original.**
3. Show that $\forall \sigma \in \mathcal{S}_n$, $\det u_\sigma = \varepsilon(\sigma)$ where ε denotes the signature.

Solution of Exercise 9 :

1. Let τ be the transposition which exchanges i and j. The matrix of u_τ in the canonical basis of $\mathbb{R}^n$ is

$$\begin{array}{c}
\begin{matrix} & & & & i & & \dots & & j & & & & \end{matrix} \\
\begin{pmatrix}
\mathbf{1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \mathbf{1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & \mathbf{1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \boxed{0} & 0 & \dots & 0 & \boxed{\mathbf{1}} & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \mathbf{1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & \mathbf{1} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \boxed{\mathbf{1}} & 0 & \dots & 0 & \boxed{0} & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \mathbf{1} & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \mathbf{1} & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \mathbf{1}
\end{pmatrix}
\end{array}$$

   By exchanging the columns i and j of the matrix of u_τ one obtains the identity matrix. Therefore $\det u_\tau = -\det I = -1$, where I denotes the identity matrix.

2. For any $\sigma, \sigma' \in \mathcal{S}_n$, one has

$$u_\sigma \circ u_{\sigma'} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = u_\sigma \begin{pmatrix} x_{\sigma'(1)} \\ \vdots \\ x_{\sigma'(n)} \end{pmatrix} = \begin{pmatrix} x_{\sigma'(\sigma(1))} \\ \vdots \\ x_{\sigma'(\sigma(n))} \end{pmatrix} = \begin{pmatrix} x_{\sigma'\circ\sigma(1)} \\ \vdots \\ x_{\sigma'\circ\sigma(n)} \end{pmatrix} = u_{\sigma'\circ\sigma} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

   Since the previous equality is satisfied for every $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$, it implies that $u_\sigma \circ u_{\sigma'} = u_{\sigma'\circ\sigma}$.

   An alternative proof is to check that u_σ sends each e_i to $e_{\sigma^{-1}(i)}$ (the basis vector whose only nonzero coordinate is the $\sigma^{-1}(i)$-th) : hence,

$$u_\sigma \circ u_{\sigma'}(e_i) = u_\sigma(e_{\sigma'^{-1}(i)}) = e_{\sigma^{-1}(\sigma'^{-1}(i))} = e_{(\sigma'\circ\sigma)^{-1}(i)} = u_{\sigma'\circ\sigma}(i).$$

3. By 2., the map which associates $u_{\sigma^{-1}}$ to a permutation σ is a group homomorphism from $\mathcal{S}_n$ into the group of invertible matrices of size (n, n), because $u_{\sigma^{-1}} \circ u_{\sigma'^{-1}} = u_{\sigma'^{-1}\circ\sigma^{-1}} = u_{(\sigma\circ\sigma')^{-1}}$. Consequently, the map which assigns to a permutation σ the number $\det u_{\sigma^{-1}}$ is a group homomorphism from $\mathcal{S}_n$ into $\{\pm 1\}$. Since the transpositions generate the group of permutations $\mathcal{S}_n$, two group homomorphisms from $\mathcal{S}_n$ to $\{\pm 1\}$ which coincide on the set of transpositions coincide on $\mathcal{S}_n$. By 1., the group homomorphism from $\mathcal{S}_n$ into $\{\pm 1\}$ which maps σ onto $\det u_{\sigma^{-1}}$ coincides with the signature on the set of transpositions, because a transposition is its own inverse. Hence $\forall \sigma \in \mathcal{S}_n$, $\det u_\sigma = \varepsilon(\sigma)$.

Exercise 10 1. Compute the eigenvalues and eigenvectors of the following matrix

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{pmatrix}.$$

2. Compute A^n for all $n \in \mathbb{N}$.

Solution of Exercise 10 :

1. One has

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & 2 & -2\\ 1 & -1-\lambda & 2\\ 1 & -3 & 4-\lambda\end{vmatrix}=-\lambda\begin{vmatrix}-1-\lambda & 2\\ -3 & 4-\lambda\end{vmatrix}-1\begin{vmatrix}2 & -2\\ -3 & 4-\lambda\end{vmatrix}+1\begin{vmatrix}2 & -2\\ -1-\lambda & 2\end{vmatrix}$$

$$=-\lambda^3+3\lambda^2-2\lambda=-\lambda(\lambda-2)(\lambda-1).$$

Therefore the eigenvalues of A are $\lambda_1=0$, $\lambda_2=2$ and $\lambda_3=1$.

A nontrivial vector in the kernel of A is given by $v_1=\begin{pmatrix}-1\\1\\1\end{pmatrix}$. Let us find a vector generating the eigenspace associated to $\lambda_2=2$. One has

$$\begin{pmatrix}A-\lambda_2 I\\ \\ I\end{pmatrix}=\begin{pmatrix}-2 & 2 & -2\\ 1 & -3 & 2\\ 1 & -3 & 2\\ \\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\xrightarrow{\substack{C_2\leftarrow C_2+C_1\\ C_3\leftarrow C_3-C_1}}\begin{pmatrix}-2 & 0 & 0\\ 1 & -2 & 1\\ 1 & -2 & 1\\ \\ 1 & 1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\xrightarrow{C_3\leftarrow 2C_3+C_2}\begin{pmatrix}-2 & 0 & 0\\ 1 & -2 & 0\\ 1 & -2 & 0\\ \\ 1 & 1 & -1\\ 0 & 1 & 1\\ 0 & 0 & 2\end{pmatrix}$$

It follows that the vector $v_2=\begin{pmatrix}-1\\1\\2\end{pmatrix}$ is a basis of the eigenspace associated to $\lambda_2=2$. Now one has

$$A-I=\begin{pmatrix}-1 & 2 & -2\\ 1 & -2 & 2\\ 1 & -3 & 3\end{pmatrix}.$$

Consequently the vector $v_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$ generates the eigenspace associated to $\lambda_3=1$.

2. Denote by f the linear application whose matrix in the canonical basis of $\mathbb{R}^3$ is A. The vectors v_1, v_2 and v_3 form a basis of $\mathbb{R}^3$. In this new basis, the matrix of f is

$$D=\begin{pmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The relation between A and D is $D=P^{-1}AP$ where $P=\begin{pmatrix}-1 & -1 & 0\\ 1 & 1 & 1\\ 1 & 2 & 1\end{pmatrix}$. The inverse of P is

$$P^{-1}=\begin{pmatrix}-1 & 1 & -1\\ 0 & -1 & 1\\ 1 & 1 & 0\end{pmatrix}.$$

Therefore, for $n>0$, we have $A^n=(PDP^{-1})(PDP^{-1})\ldots(PDP^{-1})$: cancelling all occurrences of $P^{-1}P=I$ one gets

$$A^n=PD^nP^{-1}=\begin{pmatrix}-1 & -1 & 0\\ 1 & 1 & 1\\ 1 & 2 & 1\end{pmatrix}\begin{pmatrix}0^n & 0 & 0\\ 0 & 2^n & 0\\ 0 & 0 & 1^n\end{pmatrix}\begin{pmatrix}-1 & 1 & -1\\ 0 & -1 & 1\\ 1 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 2^n & -2^n\\ 1 & -2^n+1 & 2^n\\ 1 & -2^{n+1}+1 & 2^{n+1}\end{pmatrix}.$$