University of Lille I year 2009
PC first year

Homework

Exercice 1 Apply the Gauss algorithm to the following systems :

$$\left\{\begin{array}{rcrcrcl} 3x & - & y & + & 2z & = & a \\ -x & + & 2y & - & 3z & = & b \\ x & + & 2y & + & z & = & c \end{array}\right. \qquad \left\{\begin{array}{rcrcrcl} x & + & y & + & 2z & = & 5 \\ x & - & y & - & z & = & 1 \\ x & & & + & z & = & 3 \end{array}\right.$$

Solution of Exercise 1 : Let us apply the Gauss algorithm to the first system :

$$\left\{\begin{array}{rcrcrcl} 3x & - & y & + & 2z & = & a \\ -x & + & 2y & - & 3z & = & b \\ x & + & 2y & + & z & = & c \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrcll} x & + & 2y & + & z & = & c & L_1 \leftrightarrow L_3 \\ -x & + & 2y & - & 3z & = & b & \\ 3x & - & y & + & 2z & = & a & \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcrcrcll} x & + & 2y & + & z & = & c & L_1 \\ & + & 4y & - & 2z & = & b+c & L_2 \leftarrow L_2 + L_1 \\ & - & 7y & - & z & = & a-3c & L_3 \leftarrow L_3 - 3L_1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrcll} x & + & 2y & + & z & = & c & L_1 \\ & + & y & - & \frac{1}{2}z & = & \frac{b+c}{4} & L_2 \leftarrow \frac{1}{4}L_2 \\ & - & 7y & - & z & = & a-3c & L_3 \leftarrow L_3 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcrcrcll} x & + & 2y & + & z & = & c & L_1 \\ & + & y & - & \frac{1}{2}z & = & \frac{b+c}{4} & L_2 \\ & & & - & 9z & = & \frac{1}{2}(4a-12c+7b+7c) & L_3 \leftarrow L_3 + 7L_2 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcrcrcll} x & + & 2y & + & z & = & c & L_1 \\ & + & y & - & \frac{1}{2}z & = & \frac{b+c}{4} & L_2 \\ & & & & z & = & \frac{-4a-7b+5c}{18} & L_3 \leftarrow \frac{1}{9}L_3 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rrrcl} x & & & = & c-2y-z \\ & y & & = & \frac{b+c}{4} + \frac{1}{2}z \\ & & z & = & \frac{-4a-7b+5c}{18} \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rrrcl} x & & & = & \frac{8a+5b-c}{18} \\ & y & & = & \frac{-2a+b+7c}{18} \\ & & z & = & \frac{-4a-7b+5c}{18} \end{array}\right.$$

Consequently, the first system admits a unique solution given by the triple :

$$\mathcal{S} = \left\{\left(\frac{8a+5b-c}{18}, \frac{-2a+b+7c}{18}, \frac{-4a-7b+5c}{18}\right)\right\}.$$

Let us apply the Gauss algorithm to the second system :

$$\left\{\begin{array}{rcrcrcl} x & + & y & + & 2z & = & 5 \\ x & - & y & - & z & = & 1 \\ x & & & + & z & = & 3 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrcll} x & + & y & + & 2z & = & 5 & L_1 \\ & - & 2y & - & 3z & = & -4 & L_2 \leftarrow L_2 - L_1 \\ & - & y & - & z & = & -2 & L_3 \leftarrow L_3 - L_1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcrcrcl} x & + & y & + & 2z & = & 5 \\ & & y & + & z & = & 2 \\ & & 2y & + & 3z & = & 4 \end{array}\right. \quad -L_2 \leftrightarrow -L_3 \quad \Leftrightarrow \left\{\begin{array}{rcrcrcll} x & + & y & + & 2z & = & 5 & L_1 \\ & & y & + & z & = & 2 & L_2 \\ & & & - & z & = & 0 & L_3 \leftarrow L_3 + 2L_2 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{ccccccc} x & & & & & = & 3 \\ & & y & & & = & 2 \\ & & & & z & = & 0 \end{array}\right.$$

Consequently, the second system admits a unique solution given by the triple $\{(3, 2, 0)\}$.

Exercice 2 Without trying to solve the systems, discuss the nature of the solution set of :

$$\left\{\begin{array}{ccccccc} x & + & y & - & z & = & 0 \\ x & - & y & & & = & 0 \\ x & + & y & + & z & = & 0 \end{array}\right. \quad \left\{\begin{array}{ccccccc} x & + & 3y & + & 2z & = & 1 \\ 2x & - & 2y & & & = & 2 \\ x & + & y & + & z & = & 2 \end{array}\right. \quad \left\{\begin{array}{ccccccc} x & + & 3y & + & 2z & = & 1 \\ 2x & - & 2y & & & = & 2 \\ x & + & y & + & z & = & 3 \end{array}\right.$$

Solution of Exercise 2 :
Let us apply the first steps of the Gauss algorithm to the first system :

$$\left\{\begin{array}{ccccccc} x & + & y & - & z & = & 0 \\ x & - & y & & & = & 0 \\ x & + & y & + & z & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{cccccccl} x & + & y & - & z & = & 0 & L_1 \\ & - & 2y & + & z & = & 0 & L_2 \leftarrow L_2 - L_1 \\ & & & & 2z & = & 0 & L_3 \leftarrow L_3 - L_1 \end{array}\right.$$

We obtain a Cramer system, which admits a unique solution (the rank of the system is $r = 3$, the number of equations is $p = 3$, and the number of unknowns is $n = 3$).

Let us apply the first steps of the Gauss algorithm to the second system :

$$\left\{\begin{array}{ccccccc} x & + & 3y & + & 2z & = & 1 \\ 2x & - & 2y & & & = & 2 \\ x & + & y & + & z & = & 2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{cccccccl} x & + & 3y & + & 2z & = & 1 & L_1 \\ & - & 8y & - & 4z & = & 0 & L_2 \leftarrow L_2 - 2L_1 \\ & - & 2y & - & z & = & 1 & L_3 \leftarrow L_3 - L_1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{cccccccl} x & + & 3y & + & 2z & = & 1 & L_1 \\ & - & 8y & - & 4z & = & 0 & L_2 \\ & & & & 0 & = & -4 & L_3 \leftarrow L_3 - 4L_2 \end{array}\right.$$

Since the third equation is impossible, the system admits no solution.

Let us apply the first steps of the Gauss algorithm to the third system :

$$\left\{\begin{array}{ccccccc} x & + & 3y & + & 2z & = & 1 \\ 2x & - & 2y & & & = & 2 \\ x & + & y & + & z & = & 3 \end{array}\right. \Leftrightarrow \left\{\begin{array}{cccccccl} x & + & 3y & + & 2z & = & 1 & L_1 \\ & - & 8y & - & 4z & = & 0 & L_2 \leftarrow L_2 - 2L_1 \\ & - & 2y & - & z & = & 2 & L_3 \leftarrow L_3 - L_1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{cccccccl} x & + & 3y & + & 2z & = & 1 & L_1 \\ & - & 8y & - & 4z & = & 0 & L_2 \\ & & & & 0 & = & 2 & L_3 \leftarrow L_3 - \frac{1}{4}L_2 \end{array}\right.$$

Since the third equation of the system is impossible, the system admits no solutions.

Exercice 3 Put the following systems into matrix form and solve them.

1. $\begin{cases} 2x+y+z &= 3 \\ 3x-y-2z &= 0 \\ x+y-z &= -2 \\ x+2y+z &= 1 \end{cases}$

2. $\begin{cases} x+y+z+t &= 1 \\ x-y+2z-3t &= 2 \\ 2x+4z+4t &= 3 \\ 2x+2y+3z+8t &= 2 \\ 5x+3y+9z+19t &= 6 \end{cases}$

3. $\begin{cases} 2x+y+z+t &= 1 \\ x+2y+3z+4t &= 2 \\ 3x-y-3z+2t &= 5 \\ 5y+9z-t &= -6 \end{cases}$

4. $\begin{cases} x-y+z+t &= 5 \\ 2x+3y+4z+5t &= 8 \\ 3x+y-z+t &= 7 \end{cases}$

5. $\begin{cases} x+2y+3z &= 0 \\ 2x+3y-z &= 0 \\ 3x+y+2z &= 0 \end{cases}$

Solution of Exercise 3 :

1. The extended matrix associated to the first system is :

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 3 & -1 & -2 & 0 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 1 & 1 \end{array}\right)$$

Let us apply the Gauss algorithm to this matrix :

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 3 & -1 & -2 & 0 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 1 & 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & 1 & 1 & 3 \\ 3 & -1 & -2 & 0 \\ 1 & 2 & 1 & 1 \end{array}\right) \quad \begin{array}{l} L_1 \leftarrow L_3 \\ L_2 \leftarrow L_1 \\ L_3 \leftarrow L_2 \\ L_4 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -1 & 3 & 7 \\ 0 & -4 & 1 & 6 \\ 0 & 1 & 2 & 3 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -1 & 3 & 7 \\ 0 & 0 & -11 & -22 \\ 0 & 0 & 5 & 10 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow L_3 - 4L_2 \\ L_4 \leftarrow L_4 + L_2 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -1 & 3 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow -\frac{1}{11}L_3 \\ L_4 \leftarrow \frac{1}{5}L_4 + \frac{1}{11}L_3 \end{array}$$

We obtain a triangular matrix of rank $r = 3$, hence the initial system of 4 equations with 3 unknowns admits a unique solution. Let us solve the resulting Cramer system :

$$\begin{cases} x + y - z = -2 \\ \quad - y + 3z = 7 \\ \qquad\quad z = 2 \end{cases} \Leftrightarrow \begin{cases} x = -2 - y - z \\ y = 3z - 7 \\ z = 2 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

The only solution is the point of $\mathbb{R}^3$ whose coordinates are $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

2. The extended matrix associated to the second system is :

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 & 2 \\ 2 & 0 & 4 & 4 & 3 \\ 2 & 2 & 3 & 8 & 2 \\ 5 & 3 & 9 & 19 & 6 \end{array}\right)$$

Let us apply the Gauss algorithm to this matrix :

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 & 2 \\ 2 & 0 & 4 & 4 & 3 \\ 2 & 2 & 3 & 8 & 2 \\ 5 & 3 & 9 & 19 & 6 \end{array}\right) \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 & 1 \\ 0 & -2 & 2 & 2 & 1 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & -2 & 4 & 14 & 1 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - 2L_1 \\ L_5 \leftarrow L_5 - 5L_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 & 1 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 3 & 18 & 0 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow L_3 - L_2 \\ L_4 \\ L_5 \leftarrow L_2 \end{array} \quad \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 & 1 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \\ L_4 \leftarrow L_4 - L_3 \\ L_5 \leftarrow L_5 - 3L_3 \end{array}$$

We obtain an extended matrix of rank 3. The last two lines correspond to trivially satisfied equations. Hence the initial system of 5 equations with 4 unknowns admits a line of solutions. Let us solve the equivalent triangular system :

$$\left\{\begin{array}{rcrcrcrcl} x & + & y & + & z & + & t = & 1 & \\ & - & 2y & + & z & - & 4t & = & 1 \\ & & & & z & + & 6t & = & 0 \end{array}\right.$$

To do so, let us parametrize the set of solutions by $\lambda = t \in \mathbb{R}$:

$$\Leftrightarrow \left\{\begin{array}{rcrcrcl} x & + & y & + & z & = & 1-\lambda \\ & - & 2y & + & z & = & 1+4\lambda \\ & & & & z & = & -6\lambda \\ & & & & t & = & \lambda \end{array}\right. \Leftrightarrow \left\{\begin{array}{rl} x = & 1-\lambda-y-z \\ y = & -\frac{1}{2}(1+4\lambda-z) \\ z = & -6\lambda \\ t = & \lambda \end{array}\right. \Leftrightarrow \left\{\begin{array}{rc} x = & \frac{3}{2}+10\lambda \\ y = & -\frac{1}{2}-5\lambda \\ z = & -6\lambda \\ t = & \lambda \end{array}\right.$$

Consequently, the solution of the second system is the line of $\mathbb{R}^4$ parametrized by

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ -5 \\ -6 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

3. The extended matrix associated to the third system is :

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 2 \\ 3 & -1 & -3 & 2 & 5 \\ 0 & 5 & 9 & -1 & -6 \end{array}\right)$$

Let us apply the Gauss algorithm to this matrix :

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 2 \\ 3 & -1 & -3 & 2 & 5 \\ 0 & 5 & 9 & -1 & -6 \end{array}\right) \Leftrightarrow \left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 0 & 3 & 5 & 7 & 3 \\ 0 & -5 & -9 & 1 & 7 \\ 0 & 5 & 9 & -1 & -6 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \leftarrow 2L_2 - L_1 \\ L_3 \leftarrow 2L_3 - 3L_1 \end{array}$$

Since the last two equations are incompatible, the system admits no solution.

4. The extended matrix associated to the fourth system is :

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 1 & -1 & 1 & 7 \end{array}\right)$$

Let us apply the Gauss algorithm to this matrix :

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 1 & -1 & 1 & 7 \end{array}\right) \Leftrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 0 & 5 & 2 & 3 & -2 \\ 0 & 4 & -4 & -2 & -8 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 0 & 5 & 2 & 3 & -2 \\ 0 & 0 & 14 & 11 & 16 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow 2L_3 - 5L_2 \end{array}$$

We obtain an extended matrix of rank 3. Hence the initial system of 3 equations with 4 unknowns admits a line of solutions. Let us solve the equivalent triangular system :

$$\left\{\begin{array}{ccccccccc} x & - & y & + & z & + & t = & 5 \\ & & 2y & - & 2z & - & t & = & -4 \\ & & & & 14z & + & 11t & = & 16 \end{array}\right.$$

To do so, let us parametrize the set of solutions by $\lambda = t \in \mathbb{R}$:

$$\Leftrightarrow \left\{\begin{array}{ccccccc} x & - & y & + & z & = & 5-\lambda \\ & & 2y & - & 2z & = & -4+\lambda \\ & & & & 14z & = & 16-11\lambda \\ & & & & t & = & \lambda \end{array}\right. \Leftrightarrow \left\{\begin{array}{ll} x = & 5-\lambda+y-z \\ y = & -4+\lambda+2z \\ z = & 16-11\lambda-14z \\ t = & \lambda \end{array}\right. \Leftrightarrow \left\{\begin{array}{ll} x = & 3-\frac{1}{2}\lambda \\ y = & -\frac{6}{7}-\frac{2}{7}\lambda \\ z = & -6\lambda \\ t = & \lambda \end{array}\right.$$

Consequently, the solution of the fourth system is the line of $\mathbb{R}^4$ parametrized by

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{6}{7} \\ \frac{8}{7} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{1}{2} \\ -\frac{2}{7} \\ -\frac{11}{14} \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

5. The extended matrix associated to the fifth system is :

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right)$$

Let us apply the Gauss algorithm to this matrix :

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -7 & 0 \\ 0 & -5 & -7 & 0 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -7 & 0 \\ 0 & 0 & 28 & 0 \end{array}\right) \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow L_3 - 5L_1 \end{array}$$

We obtained a Cramer system whose unique solution is $x = 0$, $y = 0$ and $z = 0$. Consequently the set of solutions of the fifth system only consists of the point $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.