BRUHAT-POISSON STRUCTURE OF THE RESTRICTED GRASSMANNIAN

A. B. TUMPACH

Abstract. In this paper, we construct a (generalized) Banach Poisson-Lie group structure on the unitary restricted Banach Lie group acting transitively on the restricted Grassmannian. A "dual" Banach Lie group consisting of (a class of) upper triangular bounded operators admits also a Poisson-Lie group structure. We show that the restricted Grassmannian inherits a Bruhat-Poisson structure from the unitary Banach Lie group, and that the action of the dual Banach Lie group on it (by "dressing transformations") is a Poisson map. This action generates the KdV hierarchy as explained in [SW85], and its orbits are the Schubert cells of the restricted Grassmannian as described in [PS88].

Keywords: restricted Grassmannian; Bruhat decomposition; Poisson manifold; coadjoint orbits; dressing transformations; Poisson-Lie groups.

Contents

Date: May 10, 2018.

1. Introduction

Poisson-Lie group and Lie bialgebras were introduced by Drinfel'd in [Dr83]. From this starting point, these notions and their relations to integrable systems were extensively studied. We refer the readers to the very well documented papers [Ko04], [STS91], [Lu90] and the references therein. For a more algebraic approach to Poisson-Lie groups and their relation to Quantum group we refer to [BHRS11]. For more details about dual pairs of Poisson manifolds we refer to [We83], applications to the study of equations coming from fluid dynamics were given in [GBV12], [GBV15] and [GBV152], and applications to geometric quantization can be found in [BW12]. The motivation to write the present paper comes mainly from the reading of [LW90], [SW85] and [PS88]. In [LW90], the Bruhat-Poisson structure of finite-dimensional Grassmannians where studied. In [SW85], the relation between the infinite-dimensional restricted Grassmannian and equations of the KdV hierarchy was established. In [PS88], the Schubert cells of the restricted Grassmannian were shown to be homogeneous spaces with respect to the natural action of some triangular group, which appears to be exactly the one that generates the KdV hierarchy in [SW85]. It is therefore natural to ask the following questions :

Question 1.1. *Does the restricted Grassmannian carry a Bruhat-Poisson structure? Can the KdV hierarchy be related to a dressing action of a Poisson-Lie group on the restricted Grassmannian?*

The difficulties to answer these questions come mainly from the following facts

- the natural action of taking the upper triangular part of some infinite-dimensional matrix does not preserve the Banach space of bounded operators, nor the Banach space of trace-class operators.
- Iwasawa decompositions may not exists in the context of infinite-dimensional Banach Lie groups (see however [Bel06] and [BN10] where some Iwasawa type factorisations where established).

Related papers on Poisson geometry in the infinite-dimensional setting are [OR03], [NST14] and [DGR15] (see Section 5). Let us mention that a hierarchy of commuting hamiltonian equations related to the restricted Grassmannian was described in [GO10]. In the aforementionned paper, the method of F. Magri was used to generate the integrals of motions. It would be interesting to explore the link between these equations and the Bruhat-Poisson structure of the restricted Grassmannian introduced in the present paper. Some integrable systems on subspaces of Hilbert-Schmidt operators were also introduced in [DO11]. There, the coinduction method suggested in [OR08] was used to construct Banach Lie-Poisson spaces obtained from the ideal of real Hilbert-Schmidt operators, and hamiltonian systems related to the k-diagonal Toda lattice were presented. Last but not least, the relation between the Bruhat-Poisson structure on the restricted Grassmannian constructed in the present paper and the Poisson-Lie group of Pseudo-Differential symbols considered in [KZ95] in relation to the Korteweg-de Vries hierarchy needs further study.

The present paper just approaches some aspects of the theory of Banach Poisson-Lie groups, and a more systematic study of the infinite-dimensional theory would be interesting. It is written to be as self-contained as possible, and we hope that our exposition enables functional-analysts, geometers and physicists to read it. However the notions of Banach manifold and fiber bundles over Banach manifolds will not be recalled and we refer the readers to [La01] for more introductory exposition.

The paper is organized as follows. In Section 2, we introduce notation used in the present paper. In Section 3, we recall the notion of duality pairing and of Manin triple, and give as example the Iwasawa Manin triple of Hilbert-Schmidt operators. In Section 4, we investigate the relation between an arbitrary duality pairing and the adjoint and coadjoint actions of a Banach Lie algebra over itself and its continuous dual. This allows to define the notion of 1-cocycle on a Banach Lie algebra $\mathfrak{g}$ with values in the Grassmann algebra of a Banach space in duality with $\mathfrak{g}$ (in the case where this Banach space is stable under the coadjoint action of $\mathfrak{g}$). We end Section 4 with Theorem 4.2 where we show that to a Banach Manin triple are naturally associated 1-cocycles of the previous type. In Section 5, we define the notion of Banach Lie bialgebras and generalize the notion of Banach Lie-Poisson spaces introduced in [OR03] to the case of an arbitrary duality pairing between two Banach Lie algebras. Then we prove the following Theorem :

Theorem 1.2 (Theorem 5.16). *Consider two Banach Lie algebras* $\left(\mathfrak{g}_+, [\cdot,\cdot]_{\mathfrak{g}_+}\right)$ *and* $\left(\mathfrak{g}_-, [\cdot,\cdot]_{\mathfrak{g}_-}\right)$ *in duality. Denote by* $\mathfrak{g}$ *the Banach space* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *with norm* $\|\cdot\|_{\mathfrak{g}} = \|\cdot\|_{\mathfrak{g}_+} + \|\cdot\|_{\mathfrak{g}_-}$. *The following assertions are equivalent.*

(1) $\mathfrak{g}_+$ *is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to* $\mathfrak{g}_-$*;*
(2) $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *is a Manin triple for the natural non-degenerate symmetric bilinear map*

$$\begin{array}{rccl} \langle\cdot,\cdot\rangle_{\mathfrak{g}} : & \mathfrak{g}\times\mathfrak{g} & \to & \mathbb{K} \\ & (x,\alpha)\times(y,\beta) & \mapsto & \langle x,\beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} + \langle y,\alpha\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}. \end{array}$$

We end Section 5 by proving that there is no Iwasawa Manin triple associated to the unitary restricted Banach Lie algebra $\mathfrak{u}_{\rm res}$ nor to its predual $\mathfrak{u}_{1,2}$. This negative result is a consequence of the fact that the triangular truncation is unbounded on the Banach space of bounded operators over a Hilbert space, as well as on the Banach space of trace class operators. In Section 6, we define a weak notion of Banach Poisson-Lie groups which relates on a weak notion of Poisson manifold, for which, following [CP12] or [NST14], we do not assume that the Poisson bracket is defined on the whole algebra of smooth functions but, contrary to [CP12] or [NST14], we neither impose the existence of hamiltonian vector fields. This weak notion is adapted to the present setting and overcomes the difficulties exposed in Section 5. In particular, examples of Banach Poisson-Lie group in our sense include the restricted unitary group $\mathrm{U}_{\rm res}(\mathcal{H})$ and the restricted triangular group $\mathrm{B}^+_{\rm res}(\mathcal{H})$. In Section 7, we show that the restricted Grassmannian viewed as homogeneous space under $\mathrm{U}_{\rm res}(\mathcal{H})$ inherites a Bruhat-Poisson structure in analogy to the finite-dimensional picture developped in [LW90]. Moreover, the natural action of the Poisson-Lie group $\mathrm{B}^+_{\rm res}(\mathcal{H})$ on the restricted Grassmannian (by "dressing transformations") is a Poisson map, and its orbits are the Schubert cells described in [PS88]. Finally the infinite-dimensional abelian subgroup of $\mathrm{B}^+_{\rm res}(\mathcal{H})$ generated by the shift induces the KdV hierarchy as explained in [SW85]. These results are summarized in the following Theorem (see Theorem 7.3, Theorem 8.5, and Theorem 8.9).

Theorem 1.3. *The restricted Grassmannian*

$$\mathrm{Gr}_{\rm res}(\mathcal{H}) = \mathrm{U}_{\rm res}(\mathcal{H})/\,\mathrm{U}(\mathcal{H}_+)\times\mathrm{U}(\mathcal{H}_-) = \mathrm{GL}_{\rm res}(\mathcal{H})/\,\mathrm{P}_{\rm res}(\mathcal{H})$$

carries a natural Poisson structure such that :

(1) *the canonical projection* $p\ :\mathrm{U}_{\rm res}(\mathcal{H})\to\mathrm{Gr}_{\rm res}(\mathcal{H})$ *is a Poisson map,*
(2) *the natural action of* $\mathrm{U}_{\rm res}(\mathcal{H})$ *on* $\mathrm{Gr}_{\rm res}(\mathcal{H})$ *by left translations is a Poisson map,*
(3) *the following right action of* $\mathrm{B}^+_{\rm res}(\mathcal{H})$ *on* $\mathrm{Gr}_{\rm res}(\mathcal{H}) = \mathrm{GL}_{\rm res}(\mathcal{H})/\,\mathrm{P}_{\rm res}(\mathcal{H})$ *is a Poisson map :*

$$\begin{array}{rcl}\mathrm{Gr}_{\rm res}(\mathcal{H})\times\mathrm{B}^+_{\rm res}(\mathcal{H}) & \to & \mathrm{Gr}_{\rm res}(\mathcal{H})\\ (g\,\mathrm{P}_{\rm res}(\mathcal{H}), b) & \mapsto & (b^{-1}g)\,\mathrm{P}_{\rm res}(\mathcal{H}).\end{array}$$

(4) *the symplectic leaves of* $\mathrm{Gr}_{\rm res}(\mathcal{H})$ *are the Schubert cells and are the orbits of* $\mathrm{B}^+_{\rm res}(\mathcal{H})$.

2. Notation

In this subsection we introduce the notation used in the present paper. In Section 2 to Section 7, $\mathcal{H}$ will refer to a general complex separable infinite-dimensional Hilbert space. The inner product in $\mathcal{H}$ will be denoted by $\langle\cdot,\cdot\rangle\ :\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ and will be complex-linear in the second variable, and conjugate-linear in the first variable. In Section 8, $\mathcal{H}$ will be specified to be the space $L^2(\mathbb{S}^1,\mathbb{C})$ of complex square-integrable functions defined almost everywhere on the unit circle $\mathbb{S}^1 = \{z\in\mathbb{C}, |z| = 1\}$ modulo the equivalence relation that identifies two functions that are equal almost everywhere. In that case, the inner product of two elements f and g in $L^2(\mathbb{S}^1,\mathbb{C})$ reads $\langle f, g\rangle = \int_{\mathbb{S}^1}\overline{f(z)}g(z)d\mu(z)$, where $d\mu(z)$ denotes the Lebesgue mesure on the circle. A tabular summarizing the notation of the paper is given at the end of the paper.

2.1. **Banach space $\mathrm{L}_\infty(\mathcal{H})$ of bounded operators over a Hilbert space $\mathcal{H}$.** Denote by $\mathrm{L}_\infty(\mathcal{H})$ the space a bounded linear maps from $\mathcal{H}$ into itself. It is a Banach space for the norm of operators $\|A\|_\infty = \sup_{\|x\|\leq 1}\|Ax\|$ and a Banach Lie algebra for the bracket given by the commutator of operators : $[A, B] = A \circ B - B \circ A$, for A, $B \in L^\infty(\mathcal{H})$. In the following, we will denote the composition $A \circ B$ of the operators A and B simply by AB.

2.2. **Schatten ideal $\mathrm{L}_2(\mathcal{H})$ of Hilbert-Schmidt operators.** A bounded operator A admits an adjoint A^* which is the bounded linear operator defined by $\langle A^*x, y\rangle = \langle x, Ay\rangle$. A positive operator is a bounded operator such that $\langle\varphi, A\varphi\rangle \geq 0$ for any $\varphi \in \mathcal{H}$. By polarization, if A is positive then $A^* = A$. The trace of a positive operator A is defined as $\mathrm{Tr}\, A = \sum_{i=1}^{+\infty}\langle\varphi_n, A\varphi_n\rangle \in [0, +\infty)$ where φ_n is any orthonormal basis of $\mathcal{H}$ (the right hand side does not depend on the choice of orthonormal basis, see Theorem 2.1 in [Sim79]). The Schatten class $\mathrm{L}_2(\mathcal{H})$ of Hilbert-Schmidt operators is the subspace of $\mathrm{L}_\infty(\mathcal{H})$ consisting of bounded operators A such that $\|A\|_2 = (\mathrm{Tr}\,(A^*A))^{\frac{1}{2}}$ is finite. It is a Banach Lie algebra for $\|\cdot\|_2$ and for the bracket given by the commutator of operators. It is also an ideal of $\mathrm{L}_\infty(\mathcal{H})$ in the sense that for any $A \in \mathrm{L}_2(\mathcal{H})$ and any $B \in \mathrm{L}_\infty(\mathcal{H})$, one has $AB \in \mathrm{L}_2(\mathcal{H})$ and $BA \in \mathrm{L}_2(\mathcal{H})$.

2.3. **Schatten ideal $\mathrm{L}_1(\mathcal{H})$ of trace class operators.** For a bounded linear operator A, the square root of A^*A is well defined, and denoted by $(A^*A)^{\frac{1}{2}}$ (see Theorem VI.9 in [RS80]). The Schatten class $\mathrm{L}_1(\mathcal{H})$ of trace class operators is the subspace of $\mathrm{L}_\infty(\mathcal{H})$ consisting of bounded operators A such that $\|A\|_1 = \mathrm{Tr}\,(A^*A)^{\frac{1}{2}}$ is finite. It is a Banach Lie algebra for $\|\cdot\|_1$ and for the bracket given by the commutator of operators. We recall that for any $A \in \mathrm{L}_1(\mathcal{H})$ (not necessarily positive), the trace of A is defined as

$$\mathrm{Tr}\, A = \sum_{i=1}^{\infty}\langle\varphi_n, A\varphi_n\rangle$$

where $\{\varphi_n\}$ is any orthonormal basis of $\mathcal{H}$ (the right hand side does not depend on the orthonormal basis, see Theorem 3.1 in [Sim79]) and that we have

$$|\mathrm{Tr}\, A| \leq \|A\|_1.$$

Moreover $\mathrm{L}_1(\mathcal{H})$ is an ideal of $\mathrm{L}_\infty(\mathcal{H})$, i.e. for any $A \in \mathrm{L}_1(\mathcal{H})$ and any $B \in \mathrm{L}_\infty(\mathcal{H})$, $AB \in \mathrm{L}_1(\mathcal{H})$ and $BA \in \mathrm{L}_1(\mathcal{H})$, and furthermore $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$. Finally for A and B in $\mathrm{L}_2(\mathcal{H})$, one has $AB \in \mathrm{L}_1(\mathcal{H})$, $BA \in \mathrm{L}_1(\mathcal{H})$, and $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$ (see Corollary 3.8 in [Sim79]).

2.4. **Restricted Banach algebra $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$ and its predual $\mathrm{L}_{1,2}(\mathcal{H})$.** Endow the infinite-dimensional separable complex Hilbert space $\mathcal{H}$ with an orthogonal decomposition into two infinite-dimensional closed subspaces : $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Denote by p_+ (resp. p_-) the orthogonal projection onto $\mathcal{H}_+$ (resp. $\mathcal{H}_-$), and set $d = i(p_+ - p_-) \in \mathrm{L}_\infty(\mathcal{H})$. The restricted Banach algebra is the Banach Lie algebra

$$\mathrm{L}_{\mathrm{res}}(\mathcal{H}) = \{A \in \mathrm{L}_\infty(\mathcal{H}), [d, A] \in \mathrm{L}_2(\mathcal{H})\} \tag{2.1}$$

for the norm $\|A\|_{\mathrm{res}} = \|A\|_\infty + \|[d, A]\|_2$ and the bracket given by the commutator of operators. A predual of $\mathrm{L}_{\mathrm{res}}$ is

$$\mathrm{L}_{1,2}(\mathcal{H}) := \{A \in \mathrm{L}_\infty(\mathcal{H}), [d, A] \in \mathrm{L}_2(\mathcal{H}), p_\pm A|_{\mathcal{H}_\pm} \in \mathrm{L}_1(\mathcal{H}_\pm)\}. \tag{2.2}$$

It is a Banach Lie algebra for the norm given by

$$\|A\|_{1,2} = \|p_+A|_{\mathcal{H}_+}\|_1 + \|p_-A|_{\mathcal{H}_-}\|_1 + \|[d,A]\|_2.$$

The duality pairing between $\mathrm{L}_{1,2}(\mathcal{H})$ and $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$ is given by

$$\begin{array}{rcll} \langle\cdot,\cdot\rangle_{\mathrm{L}_{1,2},\mathrm{L}_{\mathrm{res}}} & : & \mathrm{L}_{1,2}(\mathcal{H})\times\mathrm{L}_{\mathrm{res}}(\mathcal{H}) & \to \mathbb{C} \\ & & (A,B) & \mapsto \mathrm{Tr}(AB), \end{array}$$

where the trace is defined on $\mathrm{L}_{1,2}(\mathcal{H})$ by $\mathrm{Tr}\,A = \mathrm{Tr}\,p_+A|_{\mathcal{H}_+} + \mathrm{Tr}\,p_-A|_{\mathcal{H}_-}$ (called the *restricted trace* in [GO10]). According to Proposition 2.1 in [GO10], one has $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$ for any $A\in\mathrm{L}_{1,2}(\mathcal{H})$ and any $B\in\mathrm{L}_{\mathrm{res}}(\mathcal{H})$.

2.5. **Restricted general linear group $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$, its "predual" $\mathrm{GL}_{1,2}(\mathcal{H})$, and $\mathrm{GL}_2(\mathcal{H})$.** The general linear group of $\mathcal{H}$, denoted by $\mathrm{GL}(\mathcal{H})$ is the group consisting of bounded operators A on $\mathcal{H}$ which admit a bounded inverse, i.e. for which there exists a bounded operator A^{-1} satisfying $AA^{-1} = A^{-1}A = \mathrm{Id}$, where $\mathrm{Id} : \mathcal{H}\to\mathcal{H}$ denotes the identity operator $x\mapsto x$. The restricted general linear group, denoted by $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ is defined as

$$\mathrm{GL}_{\mathrm{res}}(\mathcal{H}) = \mathrm{GL}(\mathcal{H})\cap\mathrm{L}_{\mathrm{res}}(\mathcal{H}). \tag{2.3}$$

It is not difficult to show that $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ is closed under the operation that takes an operator $A\in\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ to its inverse $A^{-1}\in\mathrm{GL}(\mathcal{H})$, in other words that $A\in\mathrm{GL}_{\mathrm{res}}(\mathcal{H})\Rightarrow A^{-1}\in\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$. Moreover $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ has a natural Banach Lie group structure with Banach Lie algebra $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$. The Banach Lie algebra $\mathrm{L}_{1,2}(\mathcal{H})$, predual to $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$, is the Banach Lie algebra of the following Banach Lie group

$$\mathrm{GL}_{1,2}(\mathcal{H}) = \mathrm{GL}(\mathcal{H})\cap\{\mathrm{Id}+A, A\in\mathrm{L}_{1,2}(\mathcal{H})\}. \tag{2.4}$$

Similarly, the Hilbert algebra $\mathrm{L}_2(\mathcal{H})$ is the Hilbert Lie algebra of the following Hilbert Lie group :

$$\mathrm{GL}_2(\mathcal{H}) = \mathrm{GL}(\mathcal{H})\cap\{\mathrm{Id}+A, A\in\mathrm{L}_2(\mathcal{H})\}. \tag{2.5}$$

2.6. **Unitary Banach algebras $\mathfrak{u}(\mathcal{H})$, $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$, $\mathfrak{u}_{1,2}(\mathcal{H})$ and $\mathfrak{u}_2(\mathcal{H})$.** The subspace

$$\mathfrak{u}(\mathcal{H}) = \{A\in\mathrm{L}_\infty(\mathcal{H}), A^* = -A\} \tag{2.6}$$

of skew-hermitian bounded operators is a Banach Lie subalgebra of $\mathrm{L}_\infty(\mathcal{H})$. The unitary restricted algebra $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$ is the Lie subalgebra of $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$ consisting of skew-hermitian operators :

$$\mathfrak{u}_{\mathrm{res}}(\mathcal{H}) = \{A\in\mathfrak{u}(\mathcal{H}), [d,A]\in\mathrm{L}_2(\mathcal{H})\} = \mathrm{L}_{\mathrm{res}}(\mathcal{H})\cap\mathfrak{u}(\mathcal{H}). \tag{2.7}$$

By Proposition 2.1 in [BRT07], a predual of the unitary restricted algebra $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$ is the subalgebra $\mathfrak{u}_{1,2}(\mathcal{H})$ of $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$ consisting of skew-hermitian operators :

$$\mathfrak{u}_{1,2}(\mathcal{H}) := \{A\in\mathrm{L}_{1,2}(\mathcal{H}), A^* = -A\}. \tag{2.8}$$

Similarly the subspace

$$\mathfrak{u}_2(\mathcal{H}) = \{A\in\mathrm{L}_2(\mathcal{H}), A^* = -A\} \tag{2.9}$$

of skew-hermitian Hilbert-Schmidt operators is a Hilbert Lie subalgebra of $\mathrm{L}_2(\mathcal{H})$.

2.7. **Restricted unitary group** $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$**, its "predual"** $\mathrm{U}_{1,2}(\mathcal{H})$**, and** $\mathrm{U}_2(\mathcal{H})$**.** The unitary group of $\mathcal{H}$ is defined as the subgroup of $\mathrm{GL}(\mathcal{H})$ consisting of operators A such that $A^{-1} = A^*$ and is denoted by $\mathrm{U}(\mathcal{H})$. The restricted unitary group is defined by

$$\mathrm{U}_{\mathrm{res}}(\mathcal{H}) = \mathrm{GL}_{\mathrm{res}}(\mathcal{H}) \cap \mathrm{U}(\mathcal{H}). \tag{2.10}$$

It has a natural structure of real Banach Lie group with Banach Lie algebra $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$. The Banach Lie algebra $\mathfrak{u}_{1,2}(\mathcal{H})$, predual to $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$, is the Banach Lie algebra of the following Banach Lie group

$$\mathrm{U}_{1,2}(\mathcal{H}) = \mathrm{U}(\mathcal{H}) \cap \{\mathrm{Id} + A, A \in \mathrm{L}_{1,2}(\mathcal{H})\}. \tag{2.11}$$

The Hilbert Lie algebra $\mathfrak{u}_2(\mathcal{H})$ is the Hilbert Lie algebra of the following Hilbert Lie group

$$\mathrm{U}_2(\mathcal{H}) = \mathrm{U}(\mathcal{H}) \cap \{\mathrm{Id} + A, A \in \mathrm{L}_2(\mathcal{H})\}. \tag{2.12}$$

2.8. **The restricted Grassmannian** $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$**.** In the present paper, the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ denotes the set of all closed subspaces W of $\mathcal{H}$ such that the orthogonal projection $p_- : W \to \mathcal{H}_-$ is an Hilbert-Schmidt operator. For any $W \in \mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, the orthogonal projection $p_+ : W \to \mathcal{H}_+$ is a Fredholm operator whose index characterizes the connected components of $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$. The connected component of $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ containing the subspace $\mathcal{H}_+$ will be denoted by $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H})$ and consists of those subspaces W for which the orthogonal projection $p_+ : W \to \mathcal{H}_+$ has a vanishing index. The restricted Grassmannian is an homogeneous space under the restricted unitary group (see [PS88]),

$$\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) = \mathrm{U}_{\mathrm{res}}(\mathcal{H}) / \left(\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)\right),$$

and under the restricted general linear group $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$,

$$\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) = \mathrm{GL}_{\mathrm{res}}(\mathcal{H}) / \mathrm{P}_{\mathrm{res}}(\mathcal{H}),$$

where

$$\mathrm{P}_{\mathrm{res}}(\mathcal{H}) = \{A \in \mathrm{GL}_{\mathrm{res}}(\mathcal{H}), p_- A_{|\mathcal{H}_+} = 0\}. \tag{2.13}$$

It follows that $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ is a homogeneous Kähler manifold.

2.9. **Upper and lower triangular projections** T_+ **and** T_- **on Hilbert-Schmidt operators.** Endow the separable complex Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}}$ ordered according to decreasing values of n. Consider the following Banach Lie subalgebras of $\mathrm{L}_2(\mathcal{H})$

$$\mathrm{L}_2(\mathcal{H})_- = \{x \in \mathrm{L}_2(\mathcal{H}), x(|n\rangle) \in \mathrm{span}\{|m\rangle, m \leq n\}\}$$
(lower triangular operators)

$$\mathrm{L}_2(\mathcal{H})_{++} = \{x \in \mathrm{L}_2(\mathcal{H}), x(|n\rangle) \in \mathrm{span}\{|m\rangle, m > n\}\}$$
(strictly upper triangular operators).

and

$$\mathrm{L}_2(\mathcal{H})_+ = \{\alpha \in \mathrm{L}_2(\mathcal{H}), \alpha(|n\rangle) \in \mathrm{span}\{|m\rangle, m \geq n\}\}$$
(upper triangular operators)

$$\mathrm{L}_2(\mathcal{H})_{--} = \{\alpha \in \mathrm{L}_2(\mathcal{H}), \alpha(|n\rangle) \in \mathrm{span}\{|m\rangle, m < n\}\}$$
(strictly lower triangular operators).

The linear transformation T_- consisting in taking the lower triangular part of an operator with respect to the orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}}$ of $\mathcal{H}$ is called a triangular truncation or triangular projection (see [A78]) and is defined as follows :

$$\langle n, T_-(A)m\rangle = \begin{cases} \langle n, Am\rangle & \text{if} \quad n < m \\ 0 & \text{if} \quad n \geq m \end{cases} \tag{2.14}$$

Similarly, the linear transformation T_{++} consisting in taking the stricktly upper triangular part of an operator with respect to $\{|n\rangle\}_{n\in\mathbb{Z}}$ is defined as follows :

$$\langle n, T_{++}(A)m\rangle = \begin{cases} \langle n, Am\rangle & \text{if} \quad n \geq m \\ 0 & \text{if} \quad n < m \end{cases} \tag{2.15}$$

Recall that for any $A \in \mathrm{L}_2(\mathcal{H})$,

$$\|A\|_2^2 = \sum_{n,m\in\mathbb{Z}} |\langle m, An\rangle|^2,$$

hence

$$\|A\|_2^2 = \|T_-(A)\|_2^2 + \|T_{++}(A)\|_2^2.$$

It follows that both T_- and T_{++} are bounded when acting on the space of Hilbert-Schmidt operators. Consequently one has the following decompositions into sums of closed subalgebras

$$\mathrm{L}_2(\mathcal{H}) = \mathrm{L}_2(\mathcal{H})_- \oplus \mathrm{L}_2(\mathcal{H})_{++},$$

where $\mathrm{L}_2(\mathcal{H})_- = \operatorname{Ker} T_{++}$ and $\mathrm{L}_2(\mathcal{H})_{++} = \operatorname{Ker} T_-$. The linear transformation D consisting in taking the diagonal part of a linear operator is defined by

$$\langle n, D(A)m\rangle = \begin{cases} \langle n, Am\rangle & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases} \tag{2.16}$$

It is a bounded linear operator on the space of Hilbert-Schmidt operators. Denoting by $T_+ = T_{++} + D$ (resp. $T_{--} = T_- - D$) the linear transformation consisting in taking the upper triangular part (resp. strictly lower triangular part) of an operator, one has

$$\mathrm{L}_2(\mathcal{H}) = \mathrm{L}_2(\mathcal{H})_+ \oplus \mathrm{L}_2(\mathcal{H})_{--},$$

where $\mathrm{L}_2(\mathcal{H})_+ = \operatorname{Ker} T_{--}$ and $\mathrm{L}_2(\mathcal{H})_{--} = \operatorname{Ker} T_+$. In fact, the previous decompositions are orthogonal decompositions with respect to the natural Hilbert space structure of the space $\mathrm{L}_2(\mathcal{H})$ given by the trace :

$$\langle A, B\rangle_{\mathrm{L}_2} = \operatorname{Tr}(A^*B).$$

It is interesting to note that the triangular truncations T_- and T_{++} are bounded on other Schatten ideals $\mathrm{L}_p(\mathcal{H})$, for $1 < p < +\infty$. It will be of importance in the present paper that the truncation operator T_- is unbounded on the space of trace class operators $\mathrm{L}_1(\mathcal{H})$, as well as on the space of bounded operators $\mathrm{L}_\infty(\mathcal{H})$ (see Proposition 4.2 in [A78], as well as [M61], [KP70], and [GK70]).

Proposition 2.1 ([M61], [KP70], [GK70])**.** *The triangular projection T_- is bounded in the Schatten class $\mathrm{L}_p(\mathcal{H})$ if and only if $1 < p < +\infty$.*

An example of bounded operator whose triangular truncation is unbounded was given in [D88], and is defined as the limit of the following operators

$$A_n = \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & & & \frac{1}{n-1} \\ -1 & 0 & 1 & \frac{1}{2} & & & \frac{1}{n-2} \\ -\frac{1}{2} & -1 & 0 & 1 & & & \\ -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & & & \\ & & & & 0 & 1 & \frac{1}{2} \\ & & & & -1 & 0 & 1 \\ -\frac{1}{n-1} & -\frac{1}{n-2} & & & -\frac{1}{2} & -1 & 0 \end{pmatrix}.$$

As far as we know the existence and construction of a trace class operator whose triangular projection is not trace class is an open problem. We refer the reader to [Bel11] for related functional-analytic issues in the theory of Banach Lie groups.

2.10. **Triangular Banach Lie subalgebras $\mathfrak{b}_2^\pm(\mathcal{H}), \mathfrak{b}_{1,2}^\pm(\mathcal{H})$ and $\mathfrak{b}_{\rm res}^\pm$.** Let us endow $\mathcal{H}$ with an orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}}$ ordered according to decreasing values of n. Define the following triangular subalgebras of $\mathrm{L}_2(\mathcal{H})$:

$$\mathfrak{b}_2^+(\mathcal{H}) = \{\alpha \in \mathrm{L}_2(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \geq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

$$\mathfrak{b}_2^-(\mathcal{H}) = \{\alpha \in \mathrm{L}_2(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \leq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

Similarly, define the following triangular subalgebras of $\mathrm{L}_{1,2}(\mathcal{H})$ and $\mathrm{L}_{\rm res}(\mathcal{H})$, where $\mathcal{H}$ is endowed with the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ introduced in Section 2.4 :

$$\mathfrak{b}_{1,2}^+(\mathcal{H}) = \{\alpha \in \mathrm{L}_{1,2}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \geq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

$$\mathfrak{b}_{1,2}^-(\mathcal{H}) = \{\alpha \in \mathrm{L}_{1,2}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \leq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\},$$

$$\mathfrak{b}_{\rm res}^+(\mathcal{H}) = \{\alpha \in \mathrm{L}_{\rm res}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \geq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

$$\mathfrak{b}_{\rm res}^-(\mathcal{H}) = \{\alpha \in \mathrm{L}_{\rm res}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \leq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

2.11. **Triangular Banach Lie groups $\mathrm{B}_2^\pm(\mathcal{H})$, $\mathrm{B}_{1,2}^\pm(\mathcal{H})$, and $\mathrm{B}_{\rm res}^\pm(\mathcal{H})$.** To the real Hilbert Lie algebras $\mathfrak{b}_2^\pm(\mathcal{H})$ are associated the following real Hilbert Lie groups :

$$\mathrm{B}_2^\pm(\mathcal{H}) = \{\alpha \in \mathrm{GL}(\mathcal{H}) \cap \mathrm{Id} + \mathfrak{b}_2^\pm(\mathcal{H}), \text{ such that } \alpha^{-1} \in \mathrm{Id} + \mathfrak{b}_2^\pm(\mathcal{H}) \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}^{+*}, \text{for } n \in \mathbb{Z}\}.$$

To the real Banach Lie algebras $\mathfrak{b}_{1,2}^\pm(\mathcal{H})$ are associated the following real Banach Lie groups :

$$\mathrm{B}_{1,2}^\pm(\mathcal{H}) = \{\alpha \in \mathrm{GL}(\mathcal{H}) \cap \mathrm{Id} + \mathfrak{b}_{1,2}^\pm(\mathcal{H}), \text{ such that } \alpha^{-1} \in \mathrm{Id} + \mathfrak{b}_{1,2}^\pm(\mathcal{H}) \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}^{+*}, \text{for } n \in \mathbb{Z}\}.$$

To see that $\mathrm{B}_{1,2}^\pm(\mathcal{H})$ is a Banach Lie group, note that $\mathrm{B}_{1,2}^\pm(\mathcal{H})$ is an open subset of Id $+\ \mathfrak{b}_{1,2}^\pm(\mathcal{H})$, stable under group multiplication and inversion. In particular, for any $A \in \mathfrak{b}_{1,2}^\pm(\mathcal{H})$ with $\|A\|_{1,2} < 1$, and any $\alpha \in \mathrm{B}_{1,2}^\pm(\mathcal{H})$, the operator $\alpha - \alpha A$ belongs to $\mathrm{B}_{1,2}^\pm(\mathcal{H})$, since

$$(\alpha - \alpha A)^{-1} = (\mathrm{Id} - A)^{-1}\alpha^{-1},$$

and $(\mathrm{Id} - A)^{-1} = \sum_{n\in\mathbb{N}} A^n$ is a convergent series in Id $+\ \mathfrak{b}_{1,2}^\pm(\mathcal{H})$, whose limit admits strictly positive diagonal coefficients.

Similarly define the following Banach Lie groups of triangular operators :

$$\mathrm{B}_{\rm res}^\pm(\mathcal{H}) = \{\alpha \in \mathrm{GL}_{\rm res}(\mathcal{H}) \cap \mathfrak{b}_{\rm res}^\pm(\mathcal{H}) \mid \alpha^{-1} \in \mathrm{GL}_{\rm res}(\mathcal{H}) \cap \mathfrak{b}_{\rm res}^\pm(\mathcal{H}) \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}^{+*}, \text{for } n \in \mathbb{Z}\}.$$

Remark 2.2. Remark that $\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$ does not contain the shift operator $S : \mathcal{H} \to \mathcal{H}$, $|n\rangle \mapsto |n+1\rangle$ since the diagonal coefficients of any element in $\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$ are non-zero. However S belongs to the Lie algebra $\mathfrak{b}^+_{\mathrm{res}}(\mathcal{H})$, whereas S^{-1} belongs to $\mathfrak{b}^-_{\mathrm{res}}(\mathcal{H})$.

3. Manin triple of Hilbert-Schmidt operators

3.1. **Duality pairing.** Let us recall some basic notions of duality (see [AMR88], supplement 2.4.C). Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two normed vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}_1, \mathfrak{g}_2} \; : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathbb{K}$$

be a continuous bilinear map. One says that the map $\langle \cdot, \cdot \rangle_{\mathfrak{g}_1, \mathfrak{g}_2}$ is a **duality pairing** between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ if and only if it is **non-degenerate**, i.e. if the following two conditions hold :

$$(\langle x, y \rangle_{\mathfrak{g}_1, \mathfrak{g}_2} = 0, \ \forall x \in \mathfrak{g}_1) \Rightarrow y = 0 \quad \text{and} \quad (\langle x, y \rangle_{\mathfrak{g}_1, \mathfrak{g}_2} = 0, \ \forall y \in \mathfrak{g}_2) \Rightarrow x = 0.$$

One says that $\langle \cdot, \cdot \rangle_{\mathfrak{g}_1, \mathfrak{g}_2}$ is a **strong duality pairing** between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ if and only if the two continuous linear maps

$$\begin{array}{ccc} \mathfrak{g}_1 & \longrightarrow & \mathfrak{g}_2^* \\ x & \longmapsto & \langle x, \cdot \rangle_{\mathfrak{g}_1, \mathfrak{g}_2} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{g}_2 & \longrightarrow & \mathfrak{g}_1^* \\ y & \longmapsto & \langle \cdot, y \rangle_{\mathfrak{g}_1, \mathfrak{g}_2} \end{array}$$

are not only one-to-one (which is equivalent to the two conditions above) but also isomorphisms. In other words, the existence of a duality pairing between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ allows to identify $\mathfrak{g}_1$ with a subspace (not necessary closed!) of the continuous dual $\mathfrak{g}_2^*$ of $\mathfrak{g}_2$, and $\mathfrak{g}_2$ with a subspace of $\mathfrak{g}_1^*$, wheras a strong duality pairing gives isomorphisms $\mathfrak{g}_1 \simeq \mathfrak{g}_2^*$ and $\mathfrak{g}_2 \simeq \mathfrak{g}_1^*$. Therefore the existence of a strong duality pairing between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ implies that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are reflexive Banach spaces. Note that in the finite-dimensional case, a count of the dimensions shows that any duality pairing is a strong duality pairing.

Remark 3.1. Suppose that $\mathfrak{h}$ is a Banach space that injects continuously into another Banach space $\mathfrak{g}$, i.e. one has a continuous injection $\iota \; : \mathfrak{h} \hookrightarrow \mathfrak{g}$. Then one can consider two different dual spaces : the dual space $\mathfrak{h}^*$ which consists of linear forms on the Banach space $\mathfrak{h}$ which are continuous with respect to the operator norm associated to the Banach norm $\|\cdot\|_{\mathfrak{h}}$ on $\mathfrak{h}$, and the norm dual $\iota(\mathfrak{h})^*$ of the subspace $\iota(\mathfrak{h}) \subset \mathfrak{g}$ endowed with the norm $\|\cdot\|_{\mathfrak{g}}$ of $\mathfrak{g}$, consisting of continuous linear forms on the normed vector space $(\iota(\mathfrak{h}), \|\cdot\|_{\mathfrak{g}})$. Note that, since $\mathbb{R}$ is complete, $\iota(\mathfrak{h})^*$ is complete even if $\iota(\mathfrak{h})$ is not closed in $\mathfrak{g}$ (see for instance [Bre10] section 1.1). Let us compare these two duals : $\mathfrak{h}^*$ on one hand and $\iota(\mathfrak{h})^*$ on the other hand.

First note that one gets a well-defined map

$$\begin{array}{cccc} \iota^* & : & \mathfrak{g}^* & \to & \mathfrak{h}^* \\ & & f & \mapsto & f \circ \iota \end{array}$$

since $f \circ \iota$ is continuous for the operator norm induced by the norm of $\mathfrak{h}$ whenever f is continuous for the operator norm induced by the norm on $\mathfrak{g}$. Note that ι^* is surjective if and only if any continuous form on $\mathfrak{h}$ can be extended to a continuous form on $\mathfrak{g}$. On the other hand, ι^* is injective if and only if the only continuous form on $\mathfrak{g}$ that vanishes on $\iota(\mathfrak{h})$ is the zero form. Consider the following two cases, where the surjectivity (resp. injectivity) can be established.

Suppose that the range of ι is closed. Then $\iota(\mathfrak{h})$ endowed with the norm of $\mathfrak{g}$ is a Banach space. It follows that ι is a continuous bijection from the Banach space $\mathfrak{h}$ onto the Banach space $\iota(\mathfrak{h})$, therefore by the open mapping theorem, it is an isomorphism of Banach spaces (see for instance Corollary 2.7 in [Bre10]). In this case, any continuous form on $\mathfrak{h}$ is continuous for the norm of $\mathfrak{g}$ i.e. one has $\mathfrak{h}^* = \iota(\mathfrak{h})^*$. By Hahn-Banach theorem, any continuous form on $\iota(\mathfrak{h})$ can be extended to a continuous form on $\mathfrak{g}$ with the same norm (see Corollary 1.2 in [Bre10]). Therefore the dual map $\iota^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is surjective. Its kernel is the annihilator $\iota(\mathfrak{h})^0$ of $\iota(\mathfrak{h})$ and $\mathfrak{h}^*$ is isomorphic to the quotient space $\mathfrak{g}^*/\iota(\mathfrak{h})^0$. For example, the injection of the Banach space of compact operators $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$ into the Banach space of bounded operators $\mathrm{L}_\infty(\mathcal{H})$ is closed. The dual map $\iota^* : \mathrm{L}_\infty(\mathcal{H})^* \to \mathcal{K}(\mathcal{H})^* = \mathrm{L}_1(\mathcal{H})$ is surjective, and $\mathrm{L}_1(\mathcal{H})$ is isomorphic to the quotient space $\mathrm{L}_\infty(\mathcal{H})^*/\mathcal{K}(\mathcal{H})^0$. In fact, one has $\mathrm{L}_\infty(\mathcal{H})^* = \mathrm{L}_1(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H})^0$.

Now consider the case where $\iota(\mathfrak{h})$ is dense in $\mathfrak{g}$. In this case, any continuous form on $\iota(\mathfrak{h})$ extends in a unique way to a continuous form on $\mathfrak{g}$ with the same norm i.e. $\iota(\mathfrak{h})^* = \mathfrak{g}^*$. The kernel of ι^* consists of continuous maps on $\mathfrak{g}$ that vanish on the dense subspace $\iota(\mathfrak{h})$, hence is reduced to 0. In other words $\iota^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is injective (see also Corollary 1.8 in [Bre10]). For instance taking $\mathfrak{h} = \mathrm{L}_1(\mathcal{H})$ the space of trace-class operator on a Hilbert space $\mathcal{H}$ and $\mathfrak{g} = \mathrm{L}_2(\mathcal{H})$ the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$ leads to the injection $\iota^* : \mathrm{L}_2(\mathcal{H})^* = \mathrm{L}_2(\mathcal{H}) \hookrightarrow \mathrm{L}_1(\mathcal{H})^* = \mathrm{L}_\infty(\mathcal{H})$.

3.2. **Manin triples.** Let us now recall the notion of Manin triple, adapted to the context of Banach Lie algebras.

Definition 3.2. A Banach Manin triple consists of a triple of Banach Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ over a field $\mathbb{K}$ and a **non-degenerate symmetric bilinear** continuous map $\langle\cdot,\cdot\rangle_\mathfrak{g}$ on $\mathfrak{g}$ such that

(1) the scalar product $\langle\cdot,\cdot\rangle_\mathfrak{g}$ is invariant with respect to the bracket $[\cdot,\cdot]_\mathfrak{g}$ of $\mathfrak{g}$, i.e.
$$\langle [x,y]_\mathfrak{g}, z\rangle_\mathfrak{g} + \langle y, [x,z]_\mathfrak{g}\rangle_\mathfrak{g} = 0,\ \forall x,y,z \in \mathfrak{g};$$
(2) $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Banach spaces;
(3) both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Banach Lie subalgebras of $\mathfrak{g}$;
(4) both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the scalar product $\langle\cdot,\cdot\rangle_\mathfrak{g}$.

Let us make some remarks which are simple consequences of the definition of a Manin triple.

Remark 3.3. (1) Given a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, condition (2) implies that any continuous linear form α on $\mathfrak{g}$ decomposes in a continuous way as
$$\alpha = \alpha \circ p_{\mathfrak{g}_+} + \alpha \circ p_{\mathfrak{g}_-},$$
where $p_{\mathfrak{g}_+}$ (resp. $p_{\mathfrak{g}_-}$) is the continuous projection onto $\mathfrak{g}_+$ (resp. $\mathfrak{g}_-$) with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. In other words, one has a decomposition of the continuous dual $\mathfrak{g}^*$ of $\mathfrak{g}$ as
$$\mathfrak{g}^* = \mathfrak{g}_-^0 \oplus \mathfrak{g}_+^0,$$
where $\mathfrak{g}_\pm^0$ is the annihilator of $\mathfrak{g}_\pm$, i.e.
$$\mathfrak{g}_\pm^0 = \{\alpha \in \mathfrak{g}^*, \alpha(x) = 0,\ \forall x \in \mathfrak{g}_\pm\}.$$
Moreover any continuous linear form β on $\mathfrak{g}_+$ can be extended in a unique way to a continuous linear form on $\mathfrak{g}$ belonging to $\mathfrak{g}_-^0$ by $\beta \mapsto \beta \circ p_+$. It follows that one has an isomorphism
$$\mathfrak{g}_+^* = \mathfrak{g}_-^0,$$

and similarly
$$\mathfrak{g}_-^* = \mathfrak{g}_+^0.$$

(2) For a subspace $\mathfrak{h} \subset \mathfrak{g}$, we will denote by $\mathfrak{h}^\perp$ the orthogonal of $\mathfrak{h}$ with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$:
$$\mathfrak{h}^\perp = \{x \in \mathfrak{g}, \langle x, y\rangle_{\mathfrak{g}} = 0,\ \forall y \in \mathfrak{h}\}.$$
In the case where $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is a **strong** duality pairing, any continuous linear form on $\mathfrak{g}$ can be written as $\langle x,\cdot\rangle_{\mathfrak{g}}$ for some $x \in \mathfrak{g}$. In particular, for any subspace $\mathfrak{h} \subset \mathfrak{g}$, one has
$$\mathfrak{h}^0 = \mathfrak{h}^\perp.$$
Moreover, any continuous linear form β on $\mathfrak{g}_+$ can be represented as $\beta(x) = \langle x, y\rangle_{\mathfrak{g}}$ for a unique element $y \in \mathfrak{g}_-$. Therefore, in this case,
$$\mathfrak{g}_- = \mathfrak{g}_+^*$$
and similarly
$$\mathfrak{g}_+ = \mathfrak{g}_-^*.$$

3.3. **Iwasawa Manin triple of Hilbert-Schmidt operators.** The real Banach Lie algebra $\mathfrak{u}_2(\mathcal{H})$ of skew-hermitian operators in $\mathrm{L}_2(\mathcal{H})$ can be completed into a Manin triple in different ways. In this paper, we will consider the subalgebra $\mathfrak{b}_2^+(\mathcal{H})$ of $\mathrm{L}_2(\mathcal{H})$ consisting of upper triangular operators with real diagonal elements relative to the orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}}$, as well as the subalgebra $\mathfrak{b}_2^-(\mathcal{H})$ of $\mathrm{L}_2(\mathcal{H})$ consisting of lower triangular operators with real diagonal elements with respect to $\{|n\rangle\}_{n\in\mathbb{Z}}$. Recall that the basis $\{|n\rangle\}_{n\in\mathbb{Z}}$ is ordered according to decreasing values of n.
$$\mathfrak{b}_2^+(\mathcal{H}) = \{\alpha \in \mathrm{L}_2(\mathcal{H}), \alpha\left(|n\rangle\right) \in \ \mathrm{span}\{|m\rangle, m \geq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$
$$\mathfrak{b}_2^-(\mathcal{H}) = \{\alpha \in L^2(\mathcal{H}), \alpha\left(|n\rangle\right) \in \ \mathrm{span}\{|m\rangle, m \leq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$
Let us denote by $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$ the continuous bilinear map given by the imaginary part of the trace :
$$\begin{array}{rccl} \langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2} : & \mathrm{L}_2(\mathcal{H}) \times \mathrm{L}_2(\mathcal{H}) & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \Im\mathrm{Tr}\,(xy)\,. \end{array}$$

Proposition 3.4. *The triples of Hilbert Lie algebras* $(\mathrm{L}_2(\mathcal{H}), \mathfrak{u}_2(\mathcal{H}), \mathfrak{b}_2^+(\mathcal{H}))$ *and* $(\mathrm{L}_2(\mathcal{H}), \mathfrak{u}_2(\mathcal{H}), \mathfrak{b}_2^-(\mathcal{H}))$ *are real Hilbert Manin triples with respect to the pairing* $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$.

Proof. Recall that the bracket on $\mathrm{L}_2(\mathcal{H})$ is given by the commutator. For any $x, y, z \in \mathrm{L}_2(\mathcal{H})$, one has
$$\mathrm{Tr}\,([x,y]z) = \mathrm{Tr}\,(xyz - yxz) = \mathrm{Tr}\,(xyz) - \mathrm{Tr}\,(yxz) = \mathrm{Tr}\,(yzx) - \mathrm{Tr}\,(yxz) = -\mathrm{Tr}\,y[x,z],$$
where the second equality follows from the fact the both xyz and yxz are in $\mathrm{L}_1(\mathcal{H})$, and the third is justified since yz belongs to $\mathrm{L}_1(\mathcal{H})$ and x is bounded. Hence $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$ is invariant with respect to the bracket of $\mathrm{L}_2(\mathcal{H})$.

One has the following direct sum decompositions of $\mathrm{L}_2(\mathcal{H})$ into the sum of closed subalgebras
$$\mathrm{L}_2(\mathcal{H}) = \mathfrak{u}_2(\mathcal{H}) \oplus \mathfrak{b}_2^+(\mathcal{H}), \tag{3.1}$$
and
$$\mathrm{L}_2(\mathcal{H}) = \mathfrak{u}_2(\mathcal{H}) \oplus \mathfrak{b}_2^-(\mathcal{H}). \tag{3.2}$$

Note that the projections $p_{\mathfrak{u}_2^+}$ and $p_{\mathfrak{b}_2^+}$ with respect to the decomposition (3.1) onto $\mathfrak{u}_2(\mathcal{H})$ and $\mathfrak{b}_2^+(\mathcal{H})$ respectively can be expressed in terms of the orthogonal projection T_{--} and the operator D as follows :

$$p_{\mathfrak{u}_2^+}(A) = T_{--}(A) - T_{--}(A)^* + \frac{1}{2}\left[D(A) - D(A)^*\right],$$

for any $A \in \mathrm{L}_2(\mathcal{H})$, and $p_{\mathfrak{b}_2^+} = \mathrm{Id} - p_{\mathfrak{u}_p^+}$. Similarly, the projections $p_{\mathfrak{u}_2^-}$ and $p_{\mathfrak{b}_2^-}$ with respect to the decomposition (3.2) onto $\mathfrak{u}_2(\mathcal{H})$ and $\mathfrak{b}_2^-(\mathcal{H})$ respectively can be expressed in terms of the orthogonal projection T_{++} and the operator D as follows :

$$p_{\mathfrak{u}_2^-}(A) = T_{++}(A) - T_{++}(A)^* + \frac{1}{2}\left[D(A) - D(A)^*\right],$$

for any $A \in \mathrm{L}_2(\mathcal{H})$ and $p_{\mathfrak{b}_2^+} = \mathrm{Id} - p_{\mathfrak{u}_p^-}$.

Let us show that $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$ is a non-degenerate symmetric bilinear map. Consider the real Hilbert space $\mathcal{H}_{\mathbb{R}}$ generated by the orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}}$. Denote by $\Re A : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ and $\Im A : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ the real and imaginary parts of the restriction of the bounded linear operator $A \in \mathrm{L}_\infty(\mathcal{H})$ to the real Hilbert space $\mathcal{H}_{\mathbb{R}}$. Note that $A \in \mathrm{L}_2(\mathcal{H})$ if and only if $\Re A \in \mathrm{L}_2(\mathcal{H}_{\mathbb{R}})$ and $\Im A \in \mathrm{L}_2(\mathcal{H}_{\mathbb{R}})$ since

$$\|A\|_2^2 = \sum_{n,m\in\mathbb{Z}} |\langle m, An\rangle|^2 = \sum_{n,m\in\mathbb{Z}} \left(\Re\langle m, An\rangle\right)^2 + \left(\Im\langle m, An\rangle\right)^2 = \|\Re A\|_2^2 + \|\Im A\|_2^2.$$

Remark that one has

$$\Im \mathrm{Tr}\,(xy) = \mathrm{Tr}\,(\Re x \Im y + \Im x \Re y)\,,$$

for any $x \in \mathrm{L}_2(\mathcal{H})$ and any $y \in \mathrm{L}_2(\mathcal{H})$. Here x is the $\mathbb{C}$-linear extension of $\Re x + \mathrm{i}\Im x$, and y is the $\mathbb{C}$-linear extension of $\Re y + \mathrm{i}\Im y$. Since $\mathrm{L}_2(\mathcal{H}_{\mathbb{R}})$ is a Hilbert space for the inner product defined by the trace $\mathrm{Tr}\ : (A, B) \mapsto \mathrm{Tr}\,AB$, it follows that $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$ is a strong duality pairing between $\mathrm{L}_2(\mathcal{H})$ and $\mathrm{L}_2(\mathcal{H})$ viewed as real Banach spaces.

It is easy to show that $\mathfrak{u}_2(\mathcal{H}) \subset (\mathfrak{u}_2(\mathcal{H}))^\perp$, $\mathfrak{b}_2^+(\mathcal{H}) \subset \left(\mathfrak{b}_2^+(\mathcal{H})\right)^\perp$ and $\mathfrak{b}_2^-(\mathcal{H}) \subset \left(\mathfrak{b}_2^-(\mathcal{H})\right)^\perp$, in other words $\mathfrak{u}_2(\mathcal{H})$, $\mathfrak{b}_2^+(\mathcal{H})$ and $\mathfrak{b}_2^-(\mathcal{H})$ are isotropic subspaces with respect to the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$. □

Remark 3.5. Since $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$ is a strong duality pairing between $\mathrm{L}_2(\mathcal{H})$ and its dual, one has

$$\mathfrak{u}_2(\mathcal{H}) = (\mathfrak{u}_2(\mathcal{H}))^\perp = (\mathfrak{u}_2(\mathcal{H}))^0\,,$$

and

$$\mathfrak{b}_2^\pm(\mathcal{H}) = \left(\mathfrak{b}_2^\pm(\mathcal{H})\right)^\perp = \left(\mathfrak{b}_2^\pm(\mathcal{H})\right)^0.$$

Therefore

$$\left(\mathfrak{b}_2^\pm(\mathcal{H})\right)^* = L^2(\mathcal{H})/\left(\mathfrak{b}_2^\pm(\mathcal{H})\right)^0 = \mathfrak{u}_2(\mathcal{H}),$$

and

$$(\mathfrak{u}_2(\mathcal{H}))^* = L^2(\mathcal{H})/\,(\mathfrak{u}_2(\mathcal{H}))^0$$

can be identified either with $\mathfrak{b}_2^+(\mathcal{H})$ or with $\mathfrak{b}_2^-(\mathcal{H})$.

4. From Manin triples to 1-cocycles

In order to make the link between Banach Manin triples and Banach Lie bialgebra, we will need some additional notation.

4.1. **Adjoint and coadjoint actions.** Recall that a Banach Lie algebra $\mathfrak{g}_+$ acts on itself, its continuous dual $\mathfrak{g}_+^*$ and bidual $\mathfrak{g}_+^{**}$ by the adjoint and coadjoint actions

$$\begin{array}{rccl} \mathrm{ad} : & \mathfrak{g}_+ \times \mathfrak{g}_+ & \longrightarrow & \mathfrak{g}_+ \\ & (x, y) & \longmapsto & \mathrm{ad}_x y := [x, y], \end{array}$$

$$\begin{array}{rccl} -\mathrm{ad}^* : & \mathfrak{g}_+ \times \mathfrak{g}_+^* & \longrightarrow & \mathfrak{g}_+^* \\ & (x, \alpha) & \longmapsto & -\mathrm{ad}_x^* \alpha := -\alpha \circ \mathrm{ad}_x, \end{array}$$

and

$$\begin{array}{rccl} \mathrm{ad}^{**} : & \mathfrak{g}_+ \times \mathfrak{g}_+^{**} & \longrightarrow & \mathfrak{g}_+^{**} \\ & (x, \mathcal{F}) & \longmapsto & \mathrm{ad}_x^{**} \mathcal{F} := \mathcal{F} \circ \mathrm{ad}_x^*. \end{array}$$

Here the notation $\mathrm{ad}_x^* : \mathfrak{g}_+^* \to \mathfrak{g}_+^*$ means the dual map of $\mathrm{ad}_x : \mathfrak{g}_+ \to \mathfrak{g}_+$. Remark that the actions ad and ad^{**} coincide on the subspace $\mathfrak{g}_+$ of $\mathfrak{g}_+^{**}$.

To convince ourselves that the maps ad^* and ad^{**} are continuous, let us recall (see Proposition 2.2.9 in [AMR88]) that one has the following isometric isomorphisms of Banach spaces

$$\mathrm{L}(\mathfrak{g}_+^*; \mathrm{L}(\mathfrak{g}_+, \mathfrak{g}_+; \mathbb{K})) = \mathrm{L}(\mathfrak{g}_+^*, \mathfrak{g}_+, \mathfrak{g}_+; \mathbb{K}) = \mathrm{L}(\mathfrak{g}_+, \mathfrak{g}_+^*; \mathrm{L}(\mathfrak{g}_+; \mathbb{K})) = \mathrm{L}(\mathfrak{g}_+, \mathfrak{g}_+^*; \mathfrak{g}_+^*), \tag{4.1}$$

where for Banach spaces $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ and $\mathfrak{h}$, the notation $\mathrm{L}(\mathfrak{g}_1, \mathfrak{g}_2, \dots \mathfrak{g}_k; \mathfrak{h})$ stands for the Banach space of continuous k-multilinear maps from the product Banach space $\mathfrak{g}_1 \times \dots \times \mathfrak{g}_k$ to the Banach space $\mathfrak{h}$. In particular, since the map $\mathrm{ad} : \mathfrak{g}_+ \times \mathfrak{g}_+ \to \mathfrak{g}_+$ is bilinear and continuous, its dual map is continuous as a map from $\mathfrak{g}_+^* \to \mathrm{L}(\mathfrak{g}_+, \mathfrak{g}_+; \mathbb{K})$ and, following the sequence of isomorphisms in (4.1), it follows that $\mathrm{ad}^* : \mathfrak{g}_+ \times \mathfrak{g}_+^* \to \mathfrak{g}_+^*$ is continuous. Similarly, using the following isometric isomorphisms of Banach spaces

$$\mathrm{L}(\mathfrak{g}_+^{**}; \mathrm{L}(\mathfrak{g}_+, \mathfrak{g}_+^*; \mathbb{K})) = \mathrm{L}(\mathfrak{g}_+^{**}, \mathfrak{g}_+, \mathfrak{g}_+^*; \mathbb{K}) = \mathrm{L}(\mathfrak{g}_+, \mathfrak{g}_+^{**}; \mathrm{L}(\mathfrak{g}_+^*; \mathbb{K})) = \mathrm{L}(\mathfrak{g}_+, \mathfrak{g}_+^{**}; \mathfrak{g}_+^{**}),$$

it follows that $\mathrm{ad}^{**} : \mathfrak{g}_+ \times \mathfrak{g}_+^{**} \to \mathfrak{g}_+^{**}$ is continuous.

4.2. **Coadjoint action on a subspace of the dual.** Suppose that we have a continuous injection from a Banach space $\mathfrak{g}_-$ into the dual space $\mathfrak{g}_+^*$ of a Banach algebra $\mathfrak{g}_+$, in such a way that $\mathfrak{g}_-$ is stable by the coadjoint action of $\mathfrak{g}_+$ on its dual, i.e. is such that

$$\mathrm{ad}_x^* \alpha \in \mathfrak{g}_-, \ \forall x \in \mathfrak{g}_+, \forall \alpha \in \mathfrak{g}_-. \tag{4.2}$$

Then the coadjoint action $-\mathrm{ad}^* : \mathfrak{g}_+ \times \mathfrak{g}_+^* \to \mathfrak{g}_+^*$ restricts to a continuous bilinear map $-\mathrm{ad}^*_{|\mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathfrak{g}_+^*$, where $\mathfrak{g}_+ \times \mathfrak{g}_-$ is endowed with the Banach structure of the product of Banach spaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$. In other words

$$-\mathrm{ad}^*_{|\mathfrak{g}_-} \in L(\mathfrak{g}_+, \mathfrak{g}_-; \mathfrak{g}_+^*) = L(\mathfrak{g}_+; L(\mathfrak{g}_-; \mathfrak{g}_+^*)).$$

Moreover, condition (4.2) implies that $-\mathrm{ad}^*$ takes values in $\mathfrak{g}_-$, i.e. that one get a well-defined action

$$\begin{array}{rccl} -\mathrm{ad}^*_{|\mathfrak{g}_-} : & \mathfrak{g}_+ \times \mathfrak{g}_- & \longrightarrow & \mathfrak{g}_- \\ & (x, \alpha) & \longmapsto & -\mathrm{ad}_x^* \alpha := -\alpha \circ \mathrm{ad}_x. \end{array}$$

However, this action will in general not be continuous if one endows the target space with its Banach space topology. Nevertheless it is continuous if the target space is equipped with the induced topology from $\mathfrak{g}_+^*$. Under the additional assumption that $-\mathrm{ad}^*_{|\mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathfrak{g}_-$ is continuous with respect to the Banach space topologies of $\mathfrak{g}_+$ and $\mathfrak{g}_-$

(for instance in the case where $\mathfrak{g}_-$ is a closed subspace of the dual $\mathfrak{g}_+^*$), $\mathfrak{g}_+$ acts also continuously on $\mathfrak{g}_-^*$ by

$$\begin{array}{rccc}(\mathrm{ad}^*_{|\mathfrak{g}_-})^* : & \mathfrak{g}_+ \times \mathfrak{g}_-^* & \longrightarrow & \mathfrak{g}_-^* \\ & (x, \mathcal{F}) & \longmapsto & \mathcal{F} \circ \mathrm{ad}^*_x.\end{array}$$

4.3. **Adjoint representation on the space of skew-symmetric bilinear maps.** Suppose that we have a continuous injection from a Banach space $\mathfrak{g}_-$ into the dual space $\mathfrak{g}_+^*$ of a Banach algebra $\mathfrak{g}_+$ and that $\mathfrak{g}_+$ acts continously on $\mathfrak{g}_-$ by coadjoint action, i.e. suppose that $-\mathrm{ad}^*_{|\mathfrak{g}_-}$ takes values in $\mathfrak{g}_-$ and that $-\mathrm{ad}^*_{|\mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathfrak{g}_-$ is continuous. In order to simplify notation, we will write just ad^* for $\mathrm{ad}^*_{|\mathfrak{g}_-}$ and ad^{**} for $(\mathrm{ad}^*_{|\mathfrak{g}_-})^*$. Denote by $L^{r,s}(\mathfrak{g}_-, \mathfrak{g}_+; \mathbb{K})$ the space of continuous multilinear maps from $\mathfrak{g}_- \times \cdots \times \mathfrak{g}_- \times \mathfrak{g}_+ \times \cdots \times \mathfrak{g}_+$ to $\mathbb{K}$, where $\mathfrak{g}_-$ is repeated r-times and $\mathfrak{g}_+$ is repeated s-times. Since $\mathfrak{g}_+$ acts continuously by adjoint action on itself and by coadjoint action on $\mathfrak{g}_-$, one can define a continuous action of $\mathfrak{g}_+$ on $L^{r,s}(\mathfrak{g}_-, \mathfrak{g}_+; \mathbb{K})$, called also adjoint action, by

$$\begin{aligned}\mathrm{ad}^{(r,s)}_x \mathbf{t}(\alpha_1, \dots, \alpha_r, x_1, \dots, x_s) = & \sum_{i=1}^{r} \mathbf{t}(\alpha_1, \dots, \mathrm{ad}^*_x \alpha_i, \dots, \alpha_r, x_1, \dots, x_s) \\ & - \sum_{i=1}^{s} \mathbf{t}(\alpha_1, \dots, \alpha_r, x_1, \dots, \mathrm{ad}_x x_i, \dots x_s),\end{aligned}$$

where $\mathbf{t} \in L^{r,s}(\mathfrak{g}_-, \mathfrak{g}_+; \mathbb{K})$, for $i \in \{1, \dots, r\}$, $\alpha_i \in \mathfrak{g}_-$, and for $i \in \{1, \dots, s\}$, $x_i \in \mathfrak{g}_+$. In particular, the adjoint action of $\mathfrak{g}_+$ on $L^{2,0}(\mathfrak{g}_-, \mathfrak{g}_+; \mathbb{K}) = L(\mathfrak{g}_-, \mathfrak{g}_-; \mathbb{K})$ reads :

$$\mathrm{ad}^{(2,0)}_x \mathbf{t}(\alpha_1, \alpha_2) = \mathbf{t}(\mathrm{ad}^*_x \alpha_1, \alpha_2) + \mathbf{t}(\alpha_1, \mathrm{ad}^*_x \alpha_2). \tag{4.3}$$

Note that the adjoint action $\mathrm{ad}^{(2,0)}$ preserves the subspace of skew-symmetric continuous bilinear maps on $\mathfrak{g}_-$, denoted by $\Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-)$,

$$\Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-) = \{\mathbf{t} \in L(\mathfrak{g}_-, \mathfrak{g}_-; \mathbb{K}),\ \forall e_1, e_2 \in \mathfrak{g}_-, \mathbf{t}(e_1, e_2) = -\mathbf{t}(e_2, e_1)\}.$$

4.4. **Subspaces of skew-symmetric bilinear maps.** For any subspace $\mathfrak{g}_+ \subset \mathfrak{g}_-^*$, the subspace $\Lambda^2 \mathfrak{g}_+(\mathfrak{g}_-) \subset \Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-)$ refers to the subspace consisting of elements $\mathbf{t} \in \Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-)$ such that the maps $\alpha \mapsto \mathbf{t}(e_1, \alpha)$ belong to $\mathfrak{g}_+ \subset \mathfrak{g}_-^*$ for any $e_1 \in \mathfrak{g}_-$.

$$\Lambda^2 \mathfrak{g}_+(\mathfrak{g}_-) = \left\{\mathbf{t} \in \Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-),\ \forall e_1 \in \mathfrak{g}_-, (\alpha \mapsto \mathbf{t}(e_1, \alpha)) \in \mathfrak{g}_+\right\}.$$

The space $\Lambda^2 \mathfrak{g}_+(\mathfrak{g}_-)$ is therefore a space of maps acting on $\mathfrak{g}_-$, and we will keep writting $\mathfrak{g}_-$ in parenthesis in order to avoid confusions.

4.5. **1-Cocycles.** Let us recall the notion of 1-cocycle. Let G_+ be a Banach Lie group, and consider an affine action of G_+ on a Banach space V, i.e. a group morphism Φ of G_+ into the Affine group $\mathrm{Aff}(V)$ of transformations of V. Using the isomorphism $\mathrm{Aff}(V) = \mathrm{GL}(V) \rtimes V$, Φ decomposes into (φ, Θ) where $\varphi : G_+ \to \mathrm{GL}(V)$ and $\Theta : G_+ \to V$. The condition that Φ is a group morphism implies that φ is a group morphism and that Θ satisfies :

$$\Theta(gh) = \Theta(g) + \varphi(g)(\Theta(h)), \tag{4.4}$$

where $g, h \in G_+$. One says that Θ is a **1-cocycle on G_+ relative to φ**. The derivative $d\Phi$ of Φ at the unit element of G_+ is a Lie algebra morphism of the Lie algebra $\mathfrak{g}_+$ of G_+ into the Lie algebra $\mathfrak{aff}(V)$ of $\mathrm{Aff}(V)$. By the isomorphism $\mathfrak{aff}(V) = \mathfrak{gl}(V) \rtimes V$, $d\Phi$

decomposes into $(d\varphi, d\Theta)$ where $d\varphi\ : \mathfrak{g}_+ \to \mathfrak{gl}(V)$ is the Lie algebra morphism induced by φ and $d\Theta\ : \mathfrak{g}_+ \to V$ satisfies :

$$d\Theta\left([x,y]\right) = d\varphi(x)\left(d\Theta(y)\right) - d\varphi(y)\left(d\Theta(x)\right), \tag{4.5}$$

for $x, y \in \mathfrak{g}_+$. Indeed, one has for $h, g \in G_+$,

$$\Theta\left(ghg^{-1}\right) = \Theta(g) + \varphi(g)\left(\Theta(h)\right) + \varphi(ghg^{-1})\varphi(g)\left(\Theta(g^{-1})\right),$$

and for $g \in G_+$ and $y \in \mathfrak{g}_+$,

$$\Theta\left(e^{t\mathrm{Ad}(g)y}\right) = \Theta(g) + \varphi(g)\left(\Theta(e^{ty})\right) + \varphi(e^{t\mathrm{Ad}(g)y})\varphi(g)\left(\Theta(g^{-1})\right).$$

Differentiating the last equality with respect to t leads to

$$d\Theta(\mathrm{Ad}(g)y) = \varphi(g)\left(d\Theta(y)\right) + d\varphi\left(\mathrm{Ad}(g)(y)\right)\varphi(g)\left(\Theta(g^{-1})\right),$$

where Ad denotes the adjoint action of G_+ on its Lie algebra. Letting $g = e^{sx}$, with $x \in \mathfrak{g}_+$, and differentiating with respect to s gives

$$d\Theta([x,y]) = d\varphi(x)\left(d\Theta(y)\right) + d\varphi([x,y])\left(\Theta(e)\right) + d\varphi(y)d\varphi(x)\left(\Theta(e)\right) - d\varphi(y)\left(d\Theta(x)\right).$$

The cocycle identity (4.5) then follows from

$$\Theta(e) = \Theta(e\cdot e) = 0.$$

One says that $d\Theta$ is a **1-cocycle on $\mathfrak{g}$ relative to** $d\varphi$.

Examples 4.1. Let us consider in particular the Banach space $V = L(\mathfrak{g}_-, \mathfrak{g}_-; \mathbb{K})$, where $\mathfrak{g}_-$ is a Banach space that injects continuously in the dual space $\mathfrak{g}_+^*$ of a Banach algebra $\mathfrak{g}_+$, is stable under the coadjoint action of $\mathfrak{g}_+$, and such that the coadjoint action of $\mathfrak{g}_+$ on $\mathfrak{g}_-$ is continuous. A 1-cocycle θ on $\mathfrak{g}_+$ relative to the natural action $\mathrm{ad}^{(2,0)}$ of $\mathfrak{g}_+$ on $L(\mathfrak{g}_-, \mathfrak{g}_-; \mathbb{K})$ given by (4.3) is a linear map $\theta\ : \mathfrak{g}_+ \to L(\mathfrak{g}_-, \mathfrak{g}_-; \mathbb{K})$ which satisfies :

$$\theta\left([x,y]\right) \ = \mathrm{ad}^{(2,0)}_x\left(\theta(y)\right) - \mathrm{ad}^{(2,0)}_y\left(\theta(x)\right)$$

where $x, y \in \mathfrak{g}_+$. For α and β in $\mathfrak{g}_-$, one therefore has

$$\theta\left([x,y]\right)(\alpha,\beta) = \theta(y)(\mathrm{ad}^*_x\alpha,\beta) + \theta(y)(\alpha,\mathrm{ad}^*_x\beta) - \theta(x)(\mathrm{ad}^*_y\alpha,\beta) - \theta(x)(\alpha,\mathrm{ad}^*_y\beta). \tag{4.6}$$

4.6. **Manin triple and 1-cocycles.** The following proposition enable to define 1-cocycles naturally associated to a Manin triple.

Theorem 4.2. *Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple for a non-degenerated symmetric bilinear continuous map $\langle\cdot,\cdot\rangle_{\mathfrak{g}}\ : \mathfrak{g}\times\mathfrak{g} \to \mathbb{K}$. Then*

1. *The map $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ restricts to a duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}\ : \mathfrak{g}_+\times\mathfrak{g}_- \to \mathbb{K}$.*
2. *The subspace $\mathfrak{g}_+ \hookrightarrow \mathfrak{g}_-^*$ is stable under the coadjoint action of $\mathfrak{g}_-$ on $\mathfrak{g}_-^*$ and*

$$\mathrm{ad}^*_\alpha(x) = -p_{\mathfrak{g}_+}\left([\alpha, x]_{\mathfrak{g}}\right)$$

for any $x \in \mathfrak{g}_+$ and $\alpha \in \mathfrak{g}_-$. In particular,

$$\begin{array}{rccl} \mathrm{ad}^* : & \mathfrak{g}_-\times\mathfrak{g}_+ & \to & \mathfrak{g}_+ \\ & (\alpha, x) & \mapsto & -p_{\mathfrak{g}_+}\left([\alpha, x]_{\mathfrak{g}}\right) \end{array}$$

is continuous.

(3) The subspace $\mathfrak{g}_- \hookrightarrow \mathfrak{g}_+^*$ *is stable under the coadjoint action of* $\mathfrak{g}_+$ *on* $\mathfrak{g}_+^*$ *and*

$$\mathrm{ad}^*_x(\alpha) = -p_{\mathfrak{g}_-}\left([x, \alpha]_{\mathfrak{g}}\right)$$

for any $x \in \mathfrak{g}_+$ *and* $\alpha \in \mathfrak{g}_-$*. In particular,*

$$\begin{array}{rcll} \mathrm{ad}^* & : & \mathfrak{g}_+ \times \mathfrak{g}_- & \to \mathfrak{g}_- \\ & & (x, \alpha) & \mapsto -p_{\mathfrak{g}_-}\left([x, \alpha]_{\mathfrak{g}}\right) \end{array}$$

is continuous.

(4) The dual map to the bracket $[\cdot,\cdot]_{\mathfrak{g}_-}$ *restricts to a* 1*-cocycle* $\theta \ : \mathfrak{g}_+ \to \Lambda^2\mathfrak{g}_+(\mathfrak{g}_-)$ *with respect to the adjoint representation* $\mathrm{ad}^{(2,0)}$ *of* $\mathfrak{g}_+$ *on* $\Lambda^2\mathfrak{g}_+(\mathfrak{g}_-) \subset \Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$*.*

(5) The dual map to the bracket $[\cdot,\cdot]_{\mathfrak{g}_+}$ *restricts to a* 1*-cocycle* $\theta \ : \mathfrak{g}_- \to \Lambda^2\mathfrak{g}_-(\mathfrak{g}_+)$ *with respect to the adjoint representation* $\mathrm{ad}^{(2,0)}$ *of* $\mathfrak{g}_-$ *on* $\Lambda^2\mathfrak{g}_-(\mathfrak{g}_+) \subset \Lambda^2\mathfrak{g}_+^*(\mathfrak{g}_+)$*.*

Proof. (1) Let us show that the restriction of the non-degenerate bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$: $\mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ to $\mathfrak{g}_+ \times \mathfrak{g}_-$ denoted by

$$\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} \ : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$$

is a non-degenerate duality pairing between $\mathfrak{g}_+$ and $\mathfrak{g}_-$. Suppose that there exists $x \in \mathfrak{g}_+$ such that $\langle x, \alpha\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = 0$ for all $\alpha \in \mathfrak{g}_-$. Then, since $\mathfrak{g}_+$ is isotropic for $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, one has $\langle x, y\rangle_{\mathfrak{g}} = 0$ for all $y \in \mathfrak{g}$, and the non-degeneracy of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ implies that $x = 0$. The same argument apply interchanging $\mathfrak{g}_+$ and $\mathfrak{g}_-$, thus $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}$ is non-degenerate. As a consequence, one obtains two continuous injections

$$\begin{array}{rcl} \mathfrak{g}_- & \hookrightarrow & \mathfrak{g}_+^* \\ \alpha & \mapsto & \langle\cdot,\alpha\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}, \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathfrak{g}_+ & \hookrightarrow & \mathfrak{g}_-^* \\ x & \mapsto & \langle x,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}. \end{array}$$

(2)-(3) Let us show that both

$$\mathfrak{g}_+ \subset \mathfrak{g}_-^*$$

and

$$\mathfrak{g}_- \subset \mathfrak{g}_+^*$$

are stable under the coadjoint action of $\mathfrak{g}_-$ on $\mathfrak{g}_-^*$ and $\mathfrak{g}_+$ on $\mathfrak{g}_+^*$ respectively. Indeed, the invariance of the bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ with respect to the bracket $[\cdot,\cdot]_{\mathfrak{g}}$ implies that for any $x \in \mathfrak{g}_+$ and $\alpha \in \mathfrak{g}_-$,

$$\langle x, [\alpha,\cdot]_{\mathfrak{g}}\rangle_{\mathfrak{g}} = -\langle [\alpha, x]_{\mathfrak{g}}, \cdot\rangle_{\mathfrak{g}}.$$

Hence, since $\mathfrak{g}_-$ is isotropic,

$$\langle x, [\alpha,\cdot]_{\mathfrak{g}}\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = -\langle p_{\mathfrak{g}_+}\left([\alpha, x]_{\mathfrak{g}}\right), \cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-},$$

for any $x \in \mathfrak{g}_+$ and any $\alpha \in \mathfrak{g}_-$. It follows that

$$\mathrm{ad}^*_\alpha(x) = -p_{\mathfrak{g}_+}\left([\alpha, x]_{\mathfrak{g}}\right)$$

and similarly

$$\mathrm{ad}^*_x(\alpha) = -p_{\mathfrak{g}_-}\left([x, \alpha]_{\mathfrak{g}}\right)$$

for any $x \in \mathfrak{g}_+$ and $\alpha \in \mathfrak{g}_-$. The continuity of the corresponding adjoint maps follows from the continuity of the bracket $[\cdot,\cdot]_{\mathfrak{g}}$ and of the projections $p_{\mathfrak{g}_+}$ and $p_{\mathfrak{g}_-}$.

(4)-(5) Let us prove that the dual map of the Lie bracket on $\mathfrak{g}_-$ restricts to a 1-cocycle with respect to the adjoint representation of $\mathfrak{g}_+$ on $\Lambda^2\mathfrak{g}_+(\mathfrak{g}_-)$. The dual map of the bilinear map $[\cdot,\cdot]_{\mathfrak{g}_-}$ is given by

$$\begin{array}{rcccl}[\cdot,\cdot]_{\mathfrak{g}_-}^* & : & \mathfrak{g}_-^* & \longrightarrow & L(\mathfrak{g}_-,\mathfrak{g}_-;\mathbb{K}) \simeq L(\mathfrak{g}_-;\mathfrak{g}_-^*) \\ & & \mathcal{F}(\cdot) & \longmapsto & \mathcal{F}\left([\cdot,\cdot]_{\mathfrak{g}_-}\right) \mapsto \left(\alpha \mapsto \mathcal{F}\left([\alpha,\cdot]_{\mathfrak{g}_-}\right) = \mathrm{ad}^*_\alpha\mathcal{F}(\cdot)\right),\end{array}$$

and takes values in $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$. Since by (2), $\mathfrak{g}_- \subset \mathfrak{g}_+^*$ is stable under the coadjoint action of $\mathfrak{g}_+$ and since the coadjoint action $\mathrm{ad}^* : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathfrak{g}_-$ is continuous, one can consider the adjoint action of $\mathfrak{g}_+$ on $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$ defined by (4.3). Since the duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}$ induces a continuous injection $\mathfrak{g}_+ \hookrightarrow \mathfrak{g}_-^*$, one can consider the subspace $\Lambda^2\mathfrak{g}_+(\mathfrak{g}_-)$ of $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$ defined in Section 4.4. Denote by θ the restriction of $[\cdot,\cdot]_{\mathfrak{g}_-}^*$ to the subspace $\mathfrak{g}_+ \subset \mathfrak{g}_-^*$:

$$\begin{array}{rcccl}\theta & : & \mathfrak{g}_+ & \longrightarrow & L(\mathfrak{g}_-,\mathfrak{g}_-;\mathbb{K}) \simeq L(\mathfrak{g}_-;\mathfrak{g}_-^*) \\ & & x & \longmapsto & \langle x,[\cdot,\cdot]_{\mathfrak{g}_-}\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} \mapsto \left(\alpha \mapsto \langle x,[\alpha,\cdot]_{\mathfrak{g}_-}\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \mathrm{ad}^*_\alpha x(\cdot)\right).\end{array}$$

One sees immediately that the map θ takes values in $\Lambda^2\mathfrak{g}_+(\mathfrak{g}_-)$ if and only if $\mathrm{ad}^*_\alpha x \in \mathfrak{g}_+$ for any $\alpha \in \mathfrak{g}_-$ and for any $x \in \mathfrak{g}_+$, which is verified by (2). Using the fact that the duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}$ is the restriction of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is invariant with respect to the bracket $[\cdot,\cdot]_{\mathfrak{g}}$, one has

$$\langle [x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_-,\mathfrak{g}_+} = -\langle[\alpha,[x,y]],\beta\rangle_{\mathfrak{g}},$$

and the Jacobi identity verified by $[\cdot,\cdot]_{\mathfrak{g}}$ implies

$$\langle [x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_-,\mathfrak{g}_+} = -\langle[[\alpha,x],y],\beta\rangle_{\mathfrak{g}} - \langle[x,[\alpha,y]],\beta\rangle_{\mathfrak{g}}.$$

Using the decomposition

$$-[\alpha,x] = -p_{\mathfrak{g}_-}[\alpha,x] - p_{\mathfrak{g}_+}[\alpha,x] = -\mathrm{ad}^*_x\alpha + \mathrm{ad}^*_\alpha x,$$

and similarly

$$-[\alpha,y] = -p_{\mathfrak{g}_-}[\alpha,y] - p_{\mathfrak{g}_+}[\alpha,y] = -\mathrm{ad}^*_y\alpha + \mathrm{ad}^*_\alpha y,$$

one gets

$$\langle [x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \langle[\mathrm{ad}^*_\alpha x - \mathrm{ad}^*_x\alpha, y],\beta\rangle_{\mathfrak{g}} + \langle[x,\mathrm{ad}^*_\alpha y - \mathrm{ad}^*_y\alpha],\beta\rangle_{\mathfrak{g}},$$

hence

$$\begin{aligned}\langle [x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} &= \langle[\mathrm{ad}^*_\alpha x, y],\beta\rangle_{\mathfrak{g}} + \langle[x,\mathrm{ad}^*_\alpha y],\beta\rangle_{\mathfrak{g}} \\ &\quad + \langle y,[\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{g}} - \langle x,[\mathrm{ad}^*_y\alpha,\beta]\rangle_{\mathfrak{g}}.\end{aligned} \tag{4.7}$$

It follows that

$$\mathrm{ad}^*_\alpha[x,y] = [\mathrm{ad}^*_\alpha x, y] + [x,\mathrm{ad}^*_\alpha y] + \mathrm{ad}^*_{\mathrm{ad}^*_x\alpha}y - \mathrm{ad}^*_{\mathrm{ad}^*_y\alpha}x.$$

(This is exactly the formula given in [LW90] page 507, but with the opposite sign convention for the coadjoint map ad^*). On the other hand, the condition (5.4) that θ is a 1-cocycle reads :

$$\begin{aligned}\langle [x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} &= +\langle y,[\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} + \langle y,[\alpha,\mathrm{ad}^*_x\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} \\ &\quad - \langle x,[\mathrm{ad}^*_y\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} - \langle x,[\alpha,\mathrm{ad}^*_y\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}.\end{aligned} \tag{4.8}$$

The first and third terms in the RHS of (4.8) equal the last two terms in the RHS of (4.7), whereas the last term in the RHS of (4.8) reads

$$\begin{aligned}-\langle x, [\alpha, \mathrm{ad}^*_y\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} &= \langle [\alpha, x], \mathrm{ad}^*_y\beta\rangle_{\mathfrak{g}} = \langle p_{\mathfrak{g}_+}([\alpha, x]), \mathrm{ad}^*_y\beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}\\ &= -\langle \mathrm{ad}^*_\alpha x, \mathrm{ad}^*_y\beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = -\langle [y, \mathrm{ad}^*_\alpha x], \beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-},\end{aligned}$$

and similarly the second term in the RHS of (4.8) reads

$$\langle y, [\alpha, \mathrm{ad}^*_x\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \langle [x, \mathrm{ad}^*_\alpha y], \beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}.$$

Hence the equivalence between (4.8) and (4.7) follows. By interchanging the roles of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, one proves (5) in a similar way.

□

5. Banach Poisson manifolds, Banach Lie-Poisson spaces and Banach Lie bialgebras

5.1. **Definition of Banach Poisson manifolds.** The notions of Banach Poisson manifolds and Banach Lie-Poisson spaces were introduced in [OR03]. The notion of sub-Poisson manifold was introduced in [CP12] and is equivalent to the notion of generalized Poisson manifold we define below. In the case of locally convex spaces, an analoguous definition of weak Poisson manifold structure was defined in [NST14]. In the symplectic case, related notions were introduced in [DGR15] enabling the study of the orbital stability of some hamiltonian PDE's. In the present paper, we restrict ourselves to the Banach setting but generalize slightly these notions to the case where an arbitrary duality pairing is considered, and where the existence of hamiltonian vector fields is not assumed. Moreover, instead of working with subalgebras of the space of smooth functions on a Banach manifold, we will work with subbundles of the cotangent bundle (see Remark 5.2 below).

Recall that a function $f : E \to F$ between two Banach spaces is called Fréchet differentiable at $p \in E$ if there exists a bounded linear operator df_p from E to F such that

$$\lim_{x\to 0} \frac{\|f(p+x) - f(p) - df_p(x)\|_F}{\|x\|_E} = 0.$$

A function is called Fréchet differentiable on E if it is Fréchet differentiable at every $p \in E$. In that case, the Fréchet differential $df : E \to L(E, F)$ may itself be differentiable leading to the notion of $\mathcal{C}^2$ functions between the Banach spaces E and F. By induction, one can define the notion of smooth functions between two Banach spaces. A smooth real function on a Banach manifold M is a function which is smooth in any chart of M. We will denote by $\mathcal{C}^\infty(M)$ the algebra of smooth real functions on a Banach manifold M.

Definition 5.1. Consider a unital subalgebra $\mathcal{A} \subset \mathcal{C}^\infty(M)$ of smooth functions on a Banach manifold M, i.e. $\mathcal{A}$ is vector subspace of $\mathcal{C}^\infty(M)$ containing the constants and stable under pointwise multiplication. A $\mathbb{R}$-bilinear operation $\{\cdot,\cdot\} : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ is called a **Poisson bracket** on M if it satisfies :

(i) anti-symmetry : $\{f, g\} = -\{g, f\}$;
(ii) Jacobi identity : $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$;
(iii) Leibniz formula : $\{f, gh\} = \{f, g\}h + g\{f, h\}$;

Remark 5.2. (1) Note that the Leibniz rule implies that for any $f \in \mathcal{A}$, $\{f, \cdot\}$ acts by derivations on the subalgebra $\mathcal{A} \subset \mathcal{C}^\infty(M)$. When M is finite-dimensional and $\mathcal{A} = \mathcal{C}^\infty(M)$, this condition implies that $\{f, \cdot\}$ is a smooth vector field X_f on M, called the hamiltonian vector field associated to f, uniquely defined by its action on $\mathcal{C}^\infty(M)$:

$$X_f(f) = dh(X_f) = \{f, h\}.$$

It is worth noting that on a infinite-dimensional Hilbert space, there exists derivations of order greater than 1, i.e. that do not depend only on the differentials of functions (see Lemma 28.4 in [KM97], chapter VI). It follows that, contrary to the finite-dimensional case, one may not be able to associate a Poisson tensor (see Definition 5.4 below) to a given Poisson bracket. Examples of Poisson brackets not given by Poisson tensors were constructed in [BGT18].

(2) Given a covector $\xi \in T_p^*M$, it is always possible to extend it to a locally defined 1-form α with $\alpha_p = \xi$ (for instance by setting α equal to a constant in a chart around $p \in M$). However, it may not be possible to extend it to a smooth 1-form on M. It may therefore not be possible to find a smooth real function on M whose differential equals ξ at $p \in M$. The difficulty resides in defining smooth bump functions, which are, in the finite dimensional Euclidean case, usually constructed using the differentiability of the norm. In [R64], it was shown that a Banach space admits a $\mathcal{C}^1$-norm away from the origin if and only if its dual is separable. Remark that $L^\infty(\mathcal{H})$ is not separable (since it contains the nonseparable Banach space l^∞ as the space of diagonal operators). It follows that the dual of $L^\infty(\mathcal{H})$ is nonseparable (since by Theorem III.7 in [RS80], if the dual of a Banach space is separable, so is the Banach space itself). Therefore working with unital subalgebras of smooth functions on a Banach manifold modelled on $L^\infty(\mathcal{H})$ (or on $L_{\mathrm{res}}(\mathcal{H})$ and $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$) may lead to unexpected difficulties. For this reason, we will adapt the definition of Banach Poisson manifold and work with local sections of subbundles of the cotangent space. The link between unital subalgebras and subbundles of the cotangent bundle is given by next definition.

Definition 5.3. Let M be a Banach manifold and $\mathcal{A}$ be a unital subalgebra of $\mathcal{C}^\infty(M)$. The first jet of $\mathcal{A}$, denoted by $\mathrm{J}^1(\mathcal{A})$ is the subbundle of the cotangent bundle T^*M whose fiber over $p \in M$ is the space of differentials of functions in $\mathcal{A}$,

$$\mathrm{J}^1(\mathcal{A})_p = \{df_p, f \in \mathcal{A}\}.$$

Let $\mathbb{F}$ be a subbundle of the cotangent bundle T^*M, i.e. $\mathbb{F}_p$ is a subspace of T_p^*M, for every $p \in M$. Endow each fiber $\mathbb{F}_p$ with the norm of the dual space T_p^*M, $p \in M$. We will say that $\mathbb{F}$ is **in duality** with the tangent space to M if, for every $p \in M$, the natural duality pairing between T_p^*M and T_pM restricts to a duality pairing between $\mathbb{F}_p$ and T_pM, i.e. if and only if $\mathbb{F}_p$ separates points in T_pM. Note that $\mathbb{F}_p$ is complete if and only if it is closed in T_p^*M. Recall that, since $\mathbb{R}$ is complete, the dual space $\mathbb{F}_p^*$ of $\mathbb{F}_p$ is complete, even if $\mathbb{F}_p$ isn't (see for instance [Bre10] section 1.1). We will denote by $\Lambda^2\mathbb{F}^*(\mathbb{F})$ the vector bundle over M whose fiber over p is the Banach space of continuous skew-symmetric bilinear forms on the normed vector space $\mathbb{F}_p$.

Definition 5.4. Let M be a Banach manifold and $\mathbb{F}$ a subbundle of T^*M in duality with TM. A smooth section π of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a **Poisson tensor** on M with respect to $\mathbb{F}$ if :

(1) for any closed local sections α, β of $\mathbb{F}$, the differential $d\,(\pi(\alpha, \beta))$ is a local section of $\mathbb{F}$;

(2) (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta, \gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma, \alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha, \beta)\right)\right) = 0. \tag{5.1}$$

Definition 5.5. A **generalized Banach Poisson manifold** is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Remark 5.6. Taking a unital subalgebra $\mathcal{A}$ of $\mathcal{C}^\infty(M)$, $\mathbb{F} = \mathrm{J}^1(\mathcal{A})$, and $\{f, g\} = \pi(df, dg)$, our definition of generalized Banach Poisson manifold differs from the one given in [NST14] by the fact that we do not assume the existence of hamiltonian vector fields associated to functions $f \in \mathcal{A}$ (condition P3 in Definition 2.1 in [NST14]). In other words, for $f \in \mathcal{A}$, $\{f, \cdot\}$ is a derivation on $\mathcal{A} \subset \mathcal{C}^\infty(M)$ that may not –with our definition of Poisson manifold– be given by a smooth vector field on M. However, since the Poisson bracket is given by a smooth Poisson tensor, $\{f, \cdot\}$ is a smooth section of the bundle $\mathrm{J}^1(\mathcal{A})^*(\mathrm{J}^1(\mathcal{A}))$ whose fiber over $p \in M$ is the dual Banach space to the norm vector space $\mathrm{J}^1(\mathcal{A})_p$.

An important class of finite-dimensional Poisson manifolds is provided by symplectic manifolds. As we will see below, this is also the case in the Banach setting, i.e. general Banach symplectic manifolds (not necessarily strong symplectic) are particular examples of generalized Banach Poisson manifolds. Let us recall the following definitions. The exterior derivative d associates to a n-form on M a $(n+1)$-form on M. In particular, for any 2-form ω on a Banach manifold M, the exterior derivative of ω is the 3-form $d\omega$ defined by :

$$\begin{aligned} d\omega_p(X, Y, Z) = & -\omega_p([\tilde{X}, \tilde{Y}], \tilde{Z}) + \omega_p([\tilde{X}, \tilde{Z}], \tilde{Y}) - \omega_p([\tilde{Y}, \tilde{Z}], \tilde{X}) \\ & + \left\langle d_p\left(\omega(\tilde{Y}, \tilde{Z})\right), \tilde{X}\right\rangle_{T_p^*M, T_pM} - \left\langle d_p\left(\omega(\tilde{X}, \tilde{Z})\right), \tilde{Y}\right\rangle_{T_p^*M, T_pM} + \left\langle d_p\left(\omega(\tilde{X}, \tilde{Y})\right), \tilde{Z}\right\rangle_{T_p^*M, T_pM}, \end{aligned}$$

where $\tilde{X}, \tilde{Y}, \tilde{Z}$ are any smooth extensions of X, Y and $Z \in T_pM$ around $p \in M$. An expression of this formula in a chart shows that it does not depends on the extensions $\tilde{X}, \tilde{Y}, \tilde{Z}$, but only on the values of these vector fields at $p \in M$, i.e. it defines an tensor (see Proposition 3.2, chapter V in [La01]). The contraction or interior product $i_X\omega$ of a n-form ω with a vector field X is the $(n-1)$-form defined by

$$i_X\omega(Y_1, \cdots, Y_{n-1}) := \omega(X, Y_1, \cdots, Y_{n-1}).$$

The Lie derivative $\mathcal{L}_X$ with respect to a vector field X can be defined using Cartan formula

$$\mathcal{L}_X = i_X d + d\, i_X. \tag{5.2}$$

The Lie derivative, the bracket $[X, Y]$ of two vector fields X and Y, and the interior product satisfy the following relation (see Proposition 5.3, chapter V in [La01]) :

$$i_{[X,Y]} = \mathcal{L}_X i_Y - i_Y \mathcal{L}_X. \tag{5.3}$$

Let us recall the definition of a Banach (weak) symplectic manifold.

Definition 5.7. A **Banach symplectic manifold** is a Banach manifold M endowed with a 2-form $\omega \in \Gamma\left(\Lambda^2 T^*M(TM)\right)$ such that

(1) ω is non-degenerate, i.e. the map

$$\begin{array}{rccc} \omega_p^\sharp : & T_pM & \to & T_p^*M \\ & X & \mapsto & i_X\omega := \omega(X, \cdot) \end{array}$$

is injective for any $p \in M$;

(2) ω is closed, i.e. $d\omega = 0$.

Lemma 5.8. *Let (M, ω) be a Banach symplectic manifold. Consider α and β two closed local sections of $\omega^\sharp(TM)$, i.e. $d\alpha = d\beta = 0$, $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$ for some local vector fields X_α and X_β. Then*

(1) $\mathcal{L}_{X_\alpha}\omega = 0 = \mathcal{L}_{X_\beta}\omega$, in other words X_α and X_β are symplectic vector fields ;
(2) $i_{[X_\alpha, X_\beta]}\omega = -d(\omega(X_\alpha, X_\beta))$.

Proof. (1) By Cartan formula (5.2), one has $\mathcal{L}_{X_\alpha}\omega = i_{X_\alpha} d\omega + d\, i_{X_\alpha}\omega = d\, i_{X_\alpha}\omega$, since ω is closed. But by definition $i_{X_\alpha}\omega = \alpha$ is closed. Using $d \circ d = 0$ (see Supplement 6.4A in [AMR88] for a proof of this identity in the Banach context), it follows that $\mathcal{L}_{X_\alpha}\omega = 0$. Similarly $\mathcal{L}_{X_\beta}\omega = 0$.

(2) By relation (5.3), one has

$$i_{[X_\alpha, X_\beta]}\omega = \mathcal{L}_{X_\alpha} i_{X_\beta}\omega - i_{X_\beta}\mathcal{L}_{X_\alpha}\omega,$$

where the second term in the RHS vanishes by (1). Using Cartan formula, one gets

$$i_{[X_\alpha, X_\beta]}\omega = d\, i_{X_\alpha} i_{X_\beta}\omega + i_{X_\alpha} d\,(i_{X_\beta}\omega) = d\, i_{X_\alpha} i_{X_\beta}\omega = d\,(\omega(X_\beta, X_\alpha)) = -d\,(\omega(X_\alpha, X_\beta)),$$

where we have used that $i_{X_\beta}\omega = \beta$ is closed.

□

Proposition 5.9. *Any Banach symplectic manifold (M, ω) is naturally a generalized Banach Poisson manifold $(M, \mathbb{F}, \pi)$ with*

(1) $\mathbb{F} = \omega^\sharp(TM)$;
(2) π defined by

$$\begin{array}{rlcl} \pi_p & : \omega^\sharp(T_pM) \times \omega^\sharp(T_pM) & \rightarrow & \mathbb{R} \\ & (\alpha, \beta) & \mapsto & \omega(X_\alpha, X_\beta), \end{array}$$

where X_α and X_β are uniquely defined by $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$.

Proof. (1) By Lemma 5.8, for any closed local sections α and β of $\mathbb{F}$, with $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$, one has

$$d\,(\pi(\alpha, \beta)) := d\,(\omega(X_\alpha, X_\beta)) = -i_{[X_\alpha, X_\beta]}\omega,$$

hence is a local section of $\mathbb{F} = \omega^\sharp(TM)$.

(2) Let us show that π satisfies the Jacobi identity (5.1). Consider closed local sections α, β and γ of $\mathbb{F}$ and define the local vector fields X_α, X_β and X_γ by $\alpha = i_{X_\alpha}\omega$, $\beta = i_{X_\beta}\omega$ and $\gamma = i_{X_\gamma}\omega$. Using Lemma 5.8, the differential of ω satisfies

$$\begin{array}{rcl} d\omega(X_\alpha, X_\beta, X_\gamma) & = & 2\left(-\omega([X_\alpha, X_\beta], X_\gamma) + \omega([X_\alpha, X_\gamma], X_\beta) - \omega([X_\beta, X_\gamma], X_\alpha)\right) \\ & = & 2\left(\pi\left(d\left(\pi(\alpha, \beta), \gamma\right)\right)\right) + \pi\left(d\left(\pi(\gamma, \alpha)\right), \beta\right) + \pi\left(d\left(\pi(\beta, \gamma)\right), \alpha\right). \end{array}$$

Since ω is closed, the Jacobi identity (5.1) is satisfied.

□

5.2. Banach Lie-Poisson spaces.

Definition 5.10. Consider a duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$ between two Banach spaces. We will say that $\mathfrak{g}_+$ is a **Banach Lie-Poisson space with respect to** $\mathfrak{g}_-$ if $\mathfrak{g}_-$ is a Banach Lie algebra $(\mathfrak{g}_-, [\cdot,\cdot]_{\mathfrak{g}_-})$ which acts continuously on $\mathfrak{g}_+$ by coadjoint action.

Remark 5.11. A Banach Lie-Poisson space $\mathfrak{g}_+$ with respect to the dual space $\mathfrak{g}_+^*$ is a Banach Lie-Poisson space in the sense of Definition 4.1 in [OR03].

The following Theorem is a generalization of Theorem 4.2 in [OR03] to the case of an arbitrary duality pairing between two Banach spaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$ and is equivalent to Corollary 2.11 in [NST14].

Theorem 5.12. *Consider a duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$ between two Banach spaces. Suppose that $\mathfrak{g}_+$ is a Banach Lie-Poisson space with respect to $\mathfrak{g}_-$. Denote by $\mathcal{A}$ the unital subalgebra of $\mathcal{C}^\infty(\mathfrak{g}_+)$ generated by $\mathfrak{g}_-$. Define the Poisson bracket of two functions f, h in $\mathcal{A}$ by*

$$\{f,h\}(x) = \pi_x(df_x, dh_x) = \left\langle x, [df_x, dh_x]_{\mathfrak{g}_-}\right\rangle_{\mathfrak{g}_+,\mathfrak{g}_-},$$

where $x \in \mathfrak{g}_+$, and df and dh denote the Fréchet derivatives of f and h respectively. Then $(\mathfrak{g}_+, \mathrm{J}^1(\mathcal{A}), \pi)$ is a generalized Banach Poisson manifold. If h is a smooth function on $\mathfrak{g}_+$ belonging to $\mathcal{A}$, the associated hamiltonian vector field is given by

$$X_h(x) = -\mathrm{ad}^*_{dh(x)} x \in \mathfrak{g}_+.$$

Remark 5.13. One can ask whether the converse of Theorem 5.12 is true for an arbitrary duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$ between two Banach spaces. More precisely, suppose that $(\mathfrak{g}_+, \mathrm{J}^1(\mathcal{A}), \pi)$ is a generalized Banach Poisson manifold such that :

(1) $\mathcal{A}$ is the unital subalgebra of $\mathcal{C}^\infty(\mathfrak{g}_+)$ generated by $\mathfrak{g}_-$,
(2) $\mathfrak{g}_- \subset \mathcal{C}^\infty(\mathfrak{g}_+)$ is a Banach Lie algebra under the Poisson bracket operation.

Is it true that the Banach Lie algebra $(\mathfrak{g}_-, [\cdot,\cdot]_{\mathfrak{g}_-})$ acts continuously on $\mathfrak{g}_+$ by coadjoint action? If $\mathfrak{g}_+$ is closed in $\mathfrak{g}_-^*$, the answer is yes. Otherwise $\mathfrak{g}_+$ is stable by the coadjoint action of $\mathfrak{g}_-$, but the coadjoint action

$$\begin{array}{rcccl} \mathrm{ad}^* & : & \mathfrak{g}_- \times \mathfrak{g}_+ & \to & \mathfrak{g}_+ \\ & & (\alpha, x) & \mapsto & \mathrm{ad}^*_\alpha x = \pi_x(\alpha, \cdot) \end{array}$$

may not be continuous for the Banach product topology on $\mathfrak{g}_- \times \mathfrak{g}_+$ and the Banach space topology on the target space $\mathfrak{g}_+$. See also Section 4.2.

5.3. Definition of Banach Lie bialgebras.

Let us recall the definition of Lie bialgebra, adapted to the Banach setting. We refer the reader to [LW90] for the corresponding notion in the finite-dimensional case.

Definition 5.14. Let $\mathfrak{g}_+$ be a Banach Lie algebra over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and a duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}$ between $\mathfrak{g}_+$ and a normed vector space $\mathfrak{g}_-$. One says that $\mathfrak{g}_+$ is a **Banach Lie bialgebra with respect to** $\mathfrak{g}_-$ if

(1) $\mathfrak{g}_+$ acts continuously by coadjoint action on $\mathfrak{g}_-$.
(2) there is a 1-cocycle $\theta : \mathfrak{g}_+ \to \Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$ with respect to the adjoint representation $\mathrm{ad}^{(2,0)}$ of $\mathfrak{g}_+$ on $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$.

Remark 5.15. (1) The first condition in Definition 5.14 means that $\mathfrak{g}_-$ is preserved by the coadjoint action of $\mathfrak{g}_+$, i.e

$$\mathrm{ad}^*_x \mathfrak{g}_- \subset \mathfrak{g}_- \subset \mathfrak{g}_+^*$$

for any $x \in \mathfrak{g}_+$, and that the action map

$$\begin{array}{rcl} \mathfrak{g}_+ \times \mathfrak{g}_- & \to & \mathfrak{g}_- \\ (x, \alpha) & \mapsto & \mathrm{ad}^*_x \alpha \end{array}$$

is continuous. This condition is necessary in order to define the adjoint action of $\mathfrak{g}_+$ on the space $\Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-)$ of continuous skew-symmetric maps on $\mathfrak{g}_-$ by (4.3).

(2) The map θ is a 1-cocycle on $\mathfrak{g}_+$ if it satisfies :

$$\theta([x, y]) \ = \mathrm{ad}^{(2,0)}_x (\theta(y)) - \mathrm{ad}^{(2,0)}_y (\theta(x))$$

where $x, y \in \mathfrak{g}_+$. The second condition in Definition 5.14 means therefore that (see section 4.5)

$$\theta([x, y])(\alpha, \beta) = \theta(y)(\mathrm{ad}^*_x \alpha, \beta) + \theta(y)(\alpha, \mathrm{ad}^*_x \beta) - \theta(x)(\mathrm{ad}^*_y \alpha, \beta) - \theta(x)(\alpha, \mathrm{ad}^*_y \beta). \tag{5.4}$$

for any x, y in $\mathfrak{g}_+$ and any α, β in $\mathfrak{g}_-$.

(3) Let us remark that we do not assume that the cocycle θ takes values in the subspace $\Lambda^2 \mathfrak{g}_+(\mathfrak{g}_-)$ of $\Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-)$. This is related to the weak notion of Poisson manifolds given in Definition 5.5. Note also that we do not assume that $\mathfrak{g}_-$ is complete.

5.4. **Banach Lie bialgebras versus Manin triples.** In the finite-dimensional case, the notion of Lie bialgebra is equivalent to the notion of Manin triple (see for instance section 1.6 in [Ko04]). In the infinite-dimensional case the notion of Banach Lie-Poisson space comes into play.

Theorem 5.16. *Consider two Banach Lie algebras* $\left(\mathfrak{g}_+, [\cdot,\cdot]_{\mathfrak{g}_+}\right)$ *and* $\left(\mathfrak{g}_-, [\cdot,\cdot]_{\mathfrak{g}_-}\right)$ *in duality. Denote by* $\mathfrak{g}$ *the Banach space* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *with norm* $\|\cdot\|_{\mathfrak{g}} = \|\cdot\|_{\mathfrak{g}_+} + \|\cdot\|_{\mathfrak{g}_-}$*. The following assertions are equivalent.*

(1) $\mathfrak{g}_+$ *is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to* $\mathfrak{g}_-$ *with cocycle* $\theta := [\cdot,\cdot]^*_{\mathfrak{g}_-} : \mathfrak{g}_+ \to \Lambda^2 \mathfrak{g}_-^*(\mathfrak{g}_-)$*;*

(2) $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *is a Manin triple for the natural non-degenerate symmetric bilinear map*

$$\begin{array}{rccl} \langle\cdot,\cdot\rangle_{\mathfrak{g}} : & \mathfrak{g} \times \mathfrak{g} & \to & \mathbb{K} \\ & (x, \alpha) \times (y, \beta) & \mapsto & \langle x, \beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} + \langle y, \alpha\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}. \end{array}$$

(3) $\mathfrak{g}_-$ *is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to* $\mathfrak{g}_+$ *with cocycle* $\theta := [\cdot,\cdot]^*_{\mathfrak{g}_+} : \mathfrak{g}_- \to \Lambda^2 \mathfrak{g}_+^*(\mathfrak{g}_+)$*;*

Proof. (2) $\Rightarrow$ (1) follows from Theorem 4.2. Let us prove (1) $\Rightarrow$ (2). Since $\mathfrak{g}_+$ is a Banach Lie-Poisson space, $\mathfrak{g}_-$ is a Banach Lie algebra $(\mathfrak{g}_-, [\cdot,\cdot]_{\mathfrak{g}_-})$ such that the coadjoint action of $\mathfrak{g}_-$ on $\mathfrak{g}_-^*$ preserves the subspace $\mathfrak{g}_+ \subset \mathfrak{g}_-^*$ and the map

$$\begin{array}{rccl} \mathrm{ad}^* : & \mathfrak{g}_- \times \mathfrak{g}_+ & \to & \mathfrak{g}_+ \\ & (\alpha, x) & \mapsto & \mathrm{ad}^*_\alpha x, \end{array}$$

is continuous. Since $\mathfrak{g}_+$ is a Banach Lie bialgebra, the coadjoint action of $\mathfrak{g}_+$ on $\mathfrak{g}_+^*$ preserves the subspace $\mathfrak{g}_- \subset \mathfrak{g}_+^*$ and the map

$$\begin{array}{rccl} \mathrm{ad}^* : & \mathfrak{g}_+ \times \mathfrak{g}_- & \to & \mathfrak{g}_- \\ & (x, \alpha) & \mapsto & \mathrm{ad}^*_x \alpha, \end{array}$$

is continuous. Therefore the following bracket is continuous on $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$:

$$\begin{array}{rccl} [\cdot,\cdot]_{\mathfrak{g}} : & \mathfrak{g}\times\mathfrak{g} & \rightarrow & \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \\ & (x,\alpha)\times(y,\beta) & \mapsto & \left([x,y]_{\mathfrak{g}_+} + \mathrm{ad}^*_\beta x - \mathrm{ad}^*_\alpha y, \quad [\alpha,\beta]_{\mathfrak{g}_-} + \mathrm{ad}^*_y\alpha - \mathrm{ad}^*_x\beta\right). \end{array}$$

Let us show that the symmetric non-degenerate pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is invariant with respect to the bracket $[\cdot,\cdot]_{\mathfrak{g}}$. For this, we will use the fact that $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subspaces for $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. For $x\in\mathfrak{g}_+$ and $\alpha\in\mathfrak{g}_-$, one has $[x,\alpha]_{\mathfrak{g}} = (\mathrm{ad}^*_\alpha x, -\mathrm{ad}^*_x\alpha)$. Therefore, for any $x\in\mathfrak{g}_+$ and any $\alpha,\beta\in\mathfrak{g}_-$, one has

$$\begin{array}{rl} \langle[x,\alpha]_{\mathfrak{g}},\beta\rangle_{\mathfrak{g}} & = \langle\mathrm{ad}^*_\alpha x,\beta\rangle_{\mathfrak{g}} = \langle x,\mathrm{ad}_\alpha\beta\rangle_{\mathfrak{g}} = \langle x,[\alpha,\beta]_{\mathfrak{g}}\rangle_{\mathfrak{g}} \\ & = -\langle x,[\beta,\alpha]_{\mathfrak{g}}\rangle_{\mathfrak{g}} = -\langle\mathrm{ad}^*_\beta x,\alpha\rangle_{\mathfrak{g}} = \langle[\beta,x]_{\mathfrak{g}},\alpha\rangle_{\mathfrak{g}}. \end{array}$$

Similarly, for any $x,y\in\mathfrak{g}_+$ and any $\beta\in\mathfrak{g}_-$, one has

$$\langle[x,y]_{\mathfrak{g}},\beta\rangle_{\mathfrak{g}} = \langle y,\mathrm{ad}^*_x\beta\rangle_{\mathfrak{g}} = \langle y,[\beta,x]_{\mathfrak{g}}\rangle_{\mathfrak{g}} = -\langle\mathrm{ad}^*_y\beta,x\rangle_{\mathfrak{g}} = \langle[y,\beta]_{\mathfrak{g}},x\rangle_{\mathfrak{g}}.$$

By linearity, it follows that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is invariant with respect to $[\cdot,\cdot]_{\mathfrak{g}}$.

It remains to verify that $[\cdot,\cdot]_{\mathfrak{g}}$ satisfies the Jacobi identity. Let us first show that for any $x,y\in\mathfrak{g}_+$ and any $\alpha\in\mathfrak{g}_-$,

$$[\alpha,[x,y]] = [[\alpha,x],y] + [x,[\alpha,y]].$$

The dual map of the bilinear map $[\cdot,\cdot]_{\mathfrak{g}_-}$ is given by

$$\begin{array}{rcccl} [\cdot,\cdot]^*_{\mathfrak{g}_-} : & \mathfrak{g}^*_- & \longrightarrow & L(\mathfrak{g}_-,\mathfrak{g}_-;\mathbb{K}) & \simeq L(\mathfrak{g}_-;\mathfrak{g}^*_-) \\ & \mathcal{F}(\cdot) & \longmapsto & \mathcal{F}\left([\cdot,\cdot]_{\mathfrak{g}_-}\right) & \mapsto \left(\alpha\mapsto\mathcal{F}\left([\alpha,\cdot]_{\mathfrak{g}_-}\right) = \mathrm{ad}^*_\alpha\mathcal{F}(\cdot)\right). \end{array}$$

Denote by θ its restriction to the subspace $\mathfrak{g}_+$ of $\mathfrak{g}^*_-$:

$$\begin{array}{rcccl} \theta : & \mathfrak{g}_+ & \longrightarrow & L(\mathfrak{g}_-,\mathfrak{g}_-;\mathbb{K}) & \simeq L(\mathfrak{g}_-;\mathfrak{g}^*_-) \\ & x & \longmapsto & \langle x,[\cdot,\cdot]_{\mathfrak{g}_-}\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} & \mapsto \left(\alpha\mapsto\langle x,[\alpha,\cdot]_{\mathfrak{g}_-}\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \mathrm{ad}^*_\alpha x(\cdot)\right). \end{array}$$

Since $\mathfrak{g}_+$ is a Banach Lie-Poisson space, the cocycle $\theta = [\cdot,\cdot]^*_{\mathfrak{g}_-}$ restricted to $\mathfrak{g}_+\subset\mathfrak{g}^*_-$ takes values in $\Lambda^2\mathfrak{g}_+(\mathfrak{g}_-)$. The cocycle condition (5.4) reads

$$\begin{array}{rl} \langle[x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} & = +\langle y,[\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} + \langle y,[\alpha,\mathrm{ad}^*_x\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} \\ & \quad -\langle x,[\mathrm{ad}^*_y\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} - \langle x,[\alpha,\mathrm{ad}^*_y\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}, \end{array} \tag{5.5}$$

$$\begin{array}{rl} \langle[x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} & = +\langle y,[\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} + \langle y,[\alpha,\mathrm{ad}^*_x\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} \\ & \quad -\langle x,[\mathrm{ad}^*_y\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} - \langle x,[\alpha,\mathrm{ad}^*_y\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}, \end{array}$$

where $x,y\in\mathfrak{g}_+$ and $\alpha,\beta\in\mathfrak{g}_-$. Using the definition of the bracket $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and its invariance with respect to $[\cdot,\cdot]_{\mathfrak{g}}$, this is equivalent to

$$\begin{array}{rl} -\langle[\alpha,[x,y]],\beta\rangle_{\mathfrak{g}} & = -\langle[\mathrm{ad}^*_x\alpha,y],\beta\rangle_{\mathfrak{g}} - \langle[\alpha,y],\mathrm{ad}^*_x\beta\rangle_{\mathfrak{g}} \\ & \quad +\langle[\mathrm{ad}^*_y\alpha,x],\beta\rangle_{\mathfrak{g}} + \langle[\alpha,x],\mathrm{ad}^*_y\beta\rangle_{\mathfrak{g}}. \end{array}$$

Using the fact that $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subspaces for $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, one gets

$$\begin{array}{rl} -\langle[\alpha,[x,y]],\beta\rangle_{\mathfrak{g}} & = -\langle[\mathrm{ad}^*_x\alpha,y],\beta\rangle_{\mathfrak{g}} + \langle\mathrm{ad}^*_\alpha y,\mathrm{ad}^*_x\beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} \\ & \quad +\langle[\mathrm{ad}^*_y\alpha,x],\beta\rangle_{\mathfrak{g}} - \langle\mathrm{ad}^*_\alpha x,\mathrm{ad}^*_y\beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}. \end{array}$$

Using the definition of the coadjoint actions, one obtains

$$\begin{array}{rl} -\langle[\alpha,[x,y]],\beta\rangle_{\mathfrak{g}} & = -\langle[\mathrm{ad}^*_x\alpha,y],\beta\rangle_{\mathfrak{g}} + \langle[x,\mathrm{ad}^*_\alpha y],\beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} \\ & \quad +\langle[\mathrm{ad}^*_y\alpha,x],\beta\rangle_{\mathfrak{g}} - \langle[y,\mathrm{ad}^*_\alpha x],\beta\rangle_{\mathfrak{g}_+,\mathfrak{g}_-}, \end{array}$$

or, in a more compact manner,

$$-\langle[\alpha,[x,y]],\beta\rangle_{\mathfrak{g}} = \langle[\mathrm{ad}^*_\alpha x - \mathrm{ad}^*_x\alpha, y],\beta\rangle_{\mathfrak{g}} + \langle[x,\mathrm{ad}^*_\alpha y - \mathrm{ad}^*_y\alpha],\beta\rangle_{\mathfrak{g}}.$$

Using $[x,\alpha]_{\mathfrak{g}} = \mathrm{ad}^*_\alpha x - \mathrm{ad}^*_x\alpha$, and $[y,\alpha]_{\mathfrak{g}} = \mathrm{ad}^*_\alpha y - \mathrm{ad}^*_y\alpha$, one eventually gets

$$-\langle[\alpha,[x,y]],\beta\rangle_{\mathfrak{g}} = -\langle[[\alpha,x],y],\beta\rangle_{\mathfrak{g}} - \langle[x,[\alpha,y]],\beta\rangle_{\mathfrak{g}}, \tag{5.6}$$

for any $x,y\in\mathfrak{g}_+$ and any $\alpha,\beta\in\mathfrak{g}_-$. Since $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ restricts to the duality pairing between $\mathfrak{g}_+$ and $\mathfrak{g}_-$, it follows that

$$p_{\mathfrak{g}_+}[\alpha,[x,y]] = p_{\mathfrak{g}_+}[[\alpha,x],y] + p_{\mathfrak{g}_+}[x,[\alpha,y]], \tag{5.7}$$

for any $x,y\in\mathfrak{g}_+$ and any $\alpha\in\mathfrak{g}_-$. On the other hand,

$$p_{\mathfrak{g}_-}[\alpha,[x,y]] = \mathrm{ad}^*_{[x,y]}\alpha,$$

as well as

$$p_{\mathfrak{g}_-}[[\alpha,x],y] = \mathrm{ad}^*_y\mathrm{ad}^*_x\alpha,$$

and

$$p_{\mathfrak{g}_-}[x,[\alpha,y]] = -\mathrm{ad}^*_x\mathrm{ad}^*_y\alpha.$$

Using the Jacobi identity verified by the bracket in $\mathfrak{g}_+$, it follows that

$$\langle\alpha,[[x,y],z]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \langle\alpha,[x,[y,z]]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} - \langle\alpha,[y,[x,z]]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-},$$

therefore

$$p_{\mathfrak{g}_-}[\alpha,[x,y]] = p_{\mathfrak{g}_-}[[\alpha,x],y] + p_{\mathfrak{g}_-}[x,[\alpha,y]], \tag{5.8}$$

for any $x,y\in\mathfrak{g}_+$ and any $\alpha\in\mathfrak{g}_-$. Combining (5.7) and (5.8), it follows that

$$[\alpha,[x,y]] = [[\alpha,x],y] + [x,[\alpha,y]],$$

for any $x,y\in\mathfrak{g}_+$ and any $\alpha\in\mathfrak{g}_-$.

It remains to show that for any $x\in\mathfrak{g}_+$ and any $\alpha,\beta\in\mathfrak{g}_-$,

$$[x,[\alpha,\beta]] = [[x,\alpha],\beta] + [\alpha,[x,\beta]].$$

By the Jacobi identity verified by the bracket in $\mathfrak{g}_-$, one has

$$p_{\mathfrak{g}_+}[x,[\alpha,\beta]] = p_{\mathfrak{g}_+}[[x,\alpha],\beta] + p_{\mathfrak{g}_+}[\alpha,[x,\beta]]. \tag{5.9}$$

Let us show that

$$p_{\mathfrak{g}_-}[x,[\alpha,\beta]] = p_{\mathfrak{g}_-}[[x,\alpha],\beta] + p_{\mathfrak{g}_-}[\alpha,[x,\beta]],$$

for any $x\in\mathfrak{g}_+$ and any $\alpha,\beta\in\mathfrak{g}_-$. For any $x,y\in\mathfrak{g}_+$ and any $\alpha,\beta\in\mathfrak{g}_-$, one has

$$\langle y, p_{\mathfrak{g}_-}[x,[\alpha,\beta]]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = -\langle y,\mathrm{ad}^*_x[\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = -\langle[x,y],[\alpha,\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \langle[\alpha,[x,y]],\beta\rangle_{\mathfrak{g}}.$$

On the other hand, for any $x,y\in\mathfrak{g}_+$ and any $\alpha,\beta\in\mathfrak{g}_-$, one has

$$\langle y, p_{\mathfrak{g}_-}[[x,\alpha],\beta]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \langle y,[[x,\alpha],\beta]\rangle_{\mathfrak{g}} = \langle[[\alpha,x],y],\beta\rangle_{\mathfrak{g}},$$

and

$$\langle y, p_{\mathfrak{g}_-}[\alpha,[x,\beta]]\rangle_{\mathfrak{g}_+,\mathfrak{g}_-} = \langle y,[\alpha,[x,\beta]]\rangle_{\mathfrak{g}} = \langle[x,[\alpha,y]],\beta\rangle_{\mathfrak{g}}.$$

By (5.6), it follows that

$$p_{\mathfrak{g}_-}[x,[\alpha,\beta]] = p_{\mathfrak{g}_-}[[x,\alpha],\beta] + p_{\mathfrak{g}_-}[\alpha,[x,\beta]]. \tag{5.10}$$

Combining (5.9) and (5.10), it follows that

$$[x,[\alpha,\beta]] = [[x,\alpha],\beta] + [\alpha,[x,\beta]],$$

for any $x\in\mathfrak{g}_+$ and any $\alpha,\beta\in\mathfrak{g}_-$. This ends the proof of (1) $\Rightarrow$ (2). The equivalence with (3) follows by symmetry of (2). □

Remark 5.17. It is noteworthy that the cocycle condition needs only to be verified for one of the Banach Lie algebra $\mathfrak{g}_+$ or $\mathfrak{g}_-$. The following Corollary is therefore a direct consequence of the proof of Theorem 5.16.

Corollary 5.18. *Consider two Banach Lie algebras* $\left(\mathfrak{g}_+, [\cdot,\cdot]_{\mathfrak{g}_+}\right)$ *and* $\left(\mathfrak{g}_-, [\cdot,\cdot]_{\mathfrak{g}_-}\right)$ *in duality. If* $\mathfrak{g}_+$ *is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to* $\mathfrak{g}_-$*, then* $\mathfrak{g}_-$ *is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to* $\mathfrak{g}_+$*.*

5.5. **Iwasawa Banach Lie bialgebras.** Endow the separable complex Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}}$, ordered according to decreasing values of n. Let $\mathcal{H}_+$ be the complex closed subspace of $\mathcal{H}$ generated by $\{|n\rangle\}_{n\geq 0}$ and $\mathcal{H}_-$ be the complex closed subspace of $\mathcal{H}$ generated by $\{|n\rangle\}_{n<0}$. The Banach Lie algebras $L_{\mathrm{res}}(\mathcal{H})$ and $\mathrm{L}_{1,2}(\mathcal{H})$ associated to the Hilbert space decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and the corresponding unitary algebra $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$ and $\mathfrak{u}_{1,2}(\mathcal{H})$ were introduced in section 2.4. The following Banach Lie subalgebras of $L_{1,2}(\mathcal{H})$ were introduced in section 2.10 :

$$\mathfrak{b}^+_{1,2}(\mathcal{H}) = \{\alpha \in L_{1,2}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \geq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

$$\mathfrak{b}^-_{1,2}(\mathcal{H}) = \{\alpha \in L_{1,2}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \leq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

Similarly, consider the following Banach Lie subalgebras of $L_{\mathrm{res}}(\mathcal{H})$:

$$\mathfrak{b}^+_{\mathrm{res}}(\mathcal{H}) = \{\alpha \in L_{\mathrm{res}}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \geq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

$$\mathfrak{b}^-_{\mathrm{res}}(\mathcal{H}) = \{\alpha \in L_{\mathrm{res}}(\mathcal{H}), \alpha\left(|n\rangle\right) \in \text{ span}\{|m\rangle, m \leq n\} \text{ and } \langle n|\alpha|n\rangle \in \mathbb{R}, \text{for } n \in \mathbb{Z}\}.$$

Consider

$$a = \begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix} \in L_{\mathrm{res}}(\mathcal{H}),$$

where $a_{++} \in L^\infty(\mathcal{H}_+,\mathcal{H}_+)$, $a_{--} \in L^\infty(\mathcal{H}_-,\mathcal{H}_-)$, $a_{-+} \in L^2(\mathcal{H}_+,\mathcal{H}_-)$, $a_{+-} \in L^2(\mathcal{H}_-,\mathcal{H}_+)$ and similarly

$$b = \begin{pmatrix} b_{++} & b_{+-} \\ b_{-+} & b_{--} \end{pmatrix} \in \mathrm{L}_{1,2}(\mathcal{H}),$$

where $b_{++} \in L^1(\mathcal{H}_+,\mathcal{H}_+)$, $b_{--} \in L^1(\mathcal{H}_-,\mathcal{H}_-)$, $b_{-+} \in L^2(\mathcal{H}_+,\mathcal{H}_-)$, $b_{+-} \in L^2(\mathcal{H}_-,\mathcal{H}_+)$. One has

$$ab = \begin{pmatrix} a_{++}b_{++} + a_{+-}b_{-+} & a_{++}b_{+-} + a_{+-}b_{--} \\ a_{-+}b_{++} + a_{--}b_{-+} & a_{-+}b_{+-} + a_{--}b_{--} \end{pmatrix}.$$

Therefore $ab \in \mathrm{L}_{1,2}(\mathcal{H})$. The trace of ab is defined by

$$\mathrm{Tr}\, ab = \mathrm{Tr}\, a_{++}b_{++} + \mathrm{Tr}\, a_{+-}b_{-+} + \mathrm{Tr}\, a_{-+}b_{+-} + \mathrm{Tr}\, a_{--}b_{--}.$$

Recall that by Proposition 2.1 in [GO10],

$$\mathrm{Tr}\, ab = \mathrm{Tr}\, ba,$$

for every $a \in \mathrm{L}_{\mathrm{res}}(\mathcal{H})$ and $b \in \mathrm{L}_{1,2}(\mathcal{H})$. Let us denote by $\langle\cdot,\cdot\rangle_{\mathrm{L}_{\mathrm{res}},\mathrm{L}_{1,2}}$ the continuous bilinear map given by the imaginary part of the trace :

$$\begin{array}{rcccl} \langle\cdot,\cdot\rangle_{\mathrm{L}_{\mathrm{res}},\mathrm{L}_{1,2}} & : & \mathrm{L}_{\mathrm{res}}(\mathcal{H}) \times \mathrm{L}_{1,2}(\mathcal{H}) & \longrightarrow & \mathbb{R} \\ & & (x,y) & \longmapsto & \Im\mathrm{Tr}\,(xy)\,. \end{array}$$

Proposition 5.19. *The continuous bilinear map* $\langle\cdot,\cdot\rangle_{L_{\rm res},{\rm L}_{1,2}}$ *restricts to a duality pairing between* $\mathfrak{u}_{\rm res}(\mathcal{H})$ *and* $\mathfrak{b}^{\pm}_{1,2}(\mathcal{H})$ *denoted by*

$$\begin{array}{cccc}\langle\cdot,\cdot\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} & : & \mathfrak{u}_{\rm res}(\mathcal{H})\times\mathfrak{b}^{\pm}_{1,2}(\mathcal{H}) & \longrightarrow & \mathbb{R}\\ & & (x,y) & \longmapsto & \Im{\rm Tr}\,(xy)\,.\end{array}$$

Similarly the continuous bilinear map $\langle\cdot,\cdot\rangle_{L_{\rm res},{\rm L}_{1,2}}$ *restricts to a duality pairing between* $\mathfrak{b}^{\pm}_{\rm res}(\mathcal{H})$ *and* $\mathfrak{u}_{1,2}(\mathcal{H})$ *denoted by*

$$\begin{array}{cccc}\langle\cdot,\cdot\rangle_{\mathfrak{b}_{\rm res},\mathfrak{u}_{1,2}} & : & \mathfrak{b}^{\pm}_{\rm res}(\mathcal{H})\times\mathfrak{u}_{1,2}(\mathcal{H}) & \longrightarrow & \mathbb{R}\\ & & (x,y) & \longmapsto & \Im{\rm Tr}\,(xy)\,.\end{array}$$

Proof. Let us show that the map $(a,b)\mapsto\Im{\rm Tr}\,ab$ is non-degenerate for $a\in\mathfrak{u}_{\rm res}(\mathcal{H})$ and $b\in\mathfrak{b}^{+}_{1,2}(\mathcal{H})$.

Suppose that $a\in\mathfrak{u}_{\rm res}(\mathcal{H})$ is such that $\Im{\rm Tr}\,ab=0$ for any $b\in\mathfrak{b}^{+}_{1,2}(\mathcal{H})$ and let us show that a necessary vanishes. Since $\{|n\rangle\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}$ and a is bounded, it is sufficient to show that for any $n,m\in\mathbb{Z}$, $\langle n,am\rangle=0$. In fact, since a is skew-symmetric, it is enough to show that $\langle n,am\rangle=0$ for $n\leq m$. For $n\geq m$, the operator E_{nm} of rank one given by $x\mapsto\langle x,m\rangle|n\rangle$ belongs to $\mathfrak{b}^{+}_{1,2}(\mathcal{H})$. Hence for $n\geq m$, one has

$$\Im{\rm Tr}\,aE_{nm}=\Im\left(\sum_{j\in\mathbb{Z}}\langle j,m\rangle\langle j,an\rangle\right)=\Im\langle m,an\rangle=0.$$

In particular, for $m=n$, since $\langle n,an\rangle$ is purely imaginary, one has $\langle n,an\rangle=0$, $\forall n\in\mathbb{Z}$. For $n>m$, the operator iE_{nm} belongs also to $\mathfrak{b}^{+}_{1,2}(\mathcal{H})$ and

$$\Im{\rm Tr}\,aiE_{nm}=\Im\left(\sum_{j\in\mathbb{Z}}i\langle j,m\rangle\langle j,an\rangle\right)=\Re\langle m,an\rangle=0.$$

This allows to conclude that $\langle m,an\rangle=0$ for any $n,m\in\mathbb{Z}$, hence $a=0\in\mathfrak{u}_{\rm res}(\mathcal{H})$.

On the other hand, consider an element $b\in\mathfrak{b}^{+}_{1,2}(\mathcal{H})$ such that $\Im{\rm Tr}\,ab=0$ for any $a\in\mathfrak{u}_{\rm res}(\mathcal{H})$. We will show that $\langle n,bm\rangle=0$ for any $n,m\in\mathbb{Z}$ such that $n\geq m$. For $n>m$, the operator $E_{mn}-E_{nm}$ belongs to $\mathfrak{u}_{\rm res}(\mathcal{H})$, and for $n\geq m$, $iE_{mn}+iE_{nm}\in\mathfrak{u}_{\rm res}(\mathcal{H})$. Therefore for $n>m$, one has

$$\Im{\rm Tr}\,(E_{mn}-E_{nm})\,b=\Im\left(\langle n,bm\rangle-\langle m,bn\rangle\right)=\Im\langle n,bm\rangle=0,$$

and for $n\geq m$, one has

$$\Im{\rm Tr}\,(iE_{mn}+iE_{nm})\,b=\Im\left(i\langle n,bm\rangle+i\langle m,bn\rangle\right)=\Re\langle n,bm\rangle=0.$$

It follows that $\langle n,bm\rangle=0$ for all $n,m\in\mathbb{Z}$ such that $n>m$. Moreover, since $\langle n,bn\rangle\in\mathbb{R}$ for any $n\in\mathbb{Z}$, one also has $\langle n,bn\rangle=0,\forall n\in\mathbb{Z}$. Consequently $b=0$.

It follows that $\langle\cdot,\cdot\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}}$ $:\mathfrak{u}_{\rm res}(\mathcal{H})\times\mathfrak{b}^{+}_{1,2}(\mathcal{H})\to\mathbb{R}$, $(x,y)\mapsto\Im{\rm Tr}\,xy$, is non-degenerate and defines a duality pairing between $\mathfrak{u}_{\rm res}(\mathcal{H})$ and $\mathfrak{b}^{+}_{1,2}(\mathcal{H})$. One shows in a similar way that $\langle\cdot,\cdot\rangle_{{\rm L}_{\rm res},{\rm L}_{1,2}}$ induces a duality pairing between $\mathfrak{u}_{\rm res}(\mathcal{H})$ and $\mathfrak{b}^{-}_{1,2}(\mathcal{H})$, between $\mathfrak{u}_{1,2}(\mathcal{H})$ and $\mathfrak{b}^{+}_{\rm res}(\mathcal{H})$, and between $\mathfrak{u}_{1,2}(\mathcal{H})$ and $\mathfrak{b}^{-}_{\rm res}(\mathcal{H})$. □

Remark 5.20. Recall that by Proposition 2.1 in [BRT07], the dual space $\mathfrak{u}_{1,2}(\mathcal{H})^*$ can be identified with $\mathfrak{u}_{\rm res}(\mathcal{H})$, the duality pairing being given by $(a,b)\mapsto{\rm Tr}\,(ab)$. By the previous Proposition, one has a continuous injection from $\mathfrak{b}^+_{\rm res}(\mathcal{H})$ into $\mathfrak{u}_{1,2}(\mathcal{H})^*$ by $a \mapsto (b \mapsto \Im\mathrm{Tr}\,(ab))$. The corresponding injection from $\mathfrak{b}^+_{\rm res}(\mathcal{H})$ into $\mathfrak{u}_{\rm res}(\mathcal{H})$ reads :

$$\begin{array}{rcccl} \iota & : & \mathfrak{b}^+_{\rm res}(\mathcal{H}) & \hookrightarrow & \mathfrak{u}_{\rm res}(\mathcal{H}) \\ & & b & \mapsto & -\frac{i}{2}(b+b^*). \end{array}$$

The range of ι is the subspace of $\mathfrak{u}_{\rm res}(\mathcal{H})$ consisting of those $x \in \mathfrak{u}_{\rm res}(\mathcal{H})$ such that $T_-(x)$ is bounded. Recall that T_- is unbounded on $L^\infty(\mathcal{H})$, as well as on $L^1(\mathcal{H})$ (see [M61], [KP70], [GK70]), and that there exists skew-symmetric bounded operators whose triangular truncation is not bounded. Therefore ι is not be surjective.

Theorem 5.21. *The Banach Lie algebras $\mathfrak{b}^\pm_{1,2}(\mathcal{H})$ are Banach Lie bialgebras with respect to $\mathfrak{u}_{\rm res}(\mathcal{H})$. Similarly the Banach Lie algebra $\mathfrak{u}_{1,2}(\mathcal{H})$ is a Banach Lie bialgebra with respect to $\mathfrak{b}^\pm_{\rm res}(\mathcal{H})$.*

Proof. Let us show that the Lie algebra structure $[\cdot,\cdot]_{\mathfrak{u}_{\rm res}}$ on $\mathfrak{u}_{\rm res}(\mathcal{H})$ is such that

(1) $\mathfrak{b}^\pm_{1,2}(\mathcal{H})$ acts continuously by coadjoint action on $\mathfrak{u}_{\rm res}(\mathcal{H})$;
(2) the dual map $[\cdot,\cdot]^*_{\mathfrak{u}_{\rm res}} : \mathfrak{u}^*_{\rm res} \to L(\mathfrak{u}_{\rm res}, \mathfrak{u}_{\rm res}; \mathbb{K})$ to the Lie bracket $[\cdot,\cdot]_{\mathfrak{u}_{\rm res}} : \mathfrak{u}_{\rm res} \times \mathfrak{u}_{\rm res} \to \mathfrak{u}_{\rm res}$ restricts to a 1-cocycle $\theta : \mathfrak{b}^\pm_{1,2}(\mathcal{H}) \to \Lambda^2\mathfrak{u}_{\rm res}(\mathcal{H})^*(\mathfrak{u}_{\rm res}(\mathcal{H}))$ with respect to the adjoint representation $\mathrm{ad}^{(2,0)}$ of $\mathfrak{b}^\pm_{1,2}(\mathcal{H})$ on $\Lambda^2\mathfrak{u}_{\rm res}(\mathcal{H})^*(\mathfrak{u}_{\rm res}(\mathcal{H}))$.

Let us first prove (1). Since by Proposition 5.19, $\langle\cdot,\cdot\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}}$ is a duality pairing between $\mathfrak{u}_{\rm res}(\mathcal{H})$ and $\mathfrak{b}^\pm_{1,2}(\mathcal{H})$, the Banach space $\mathfrak{u}_{\rm res}(\mathcal{H})$ is a subspace in the continuous dual of $\mathfrak{b}^\pm_{1,2}(\mathcal{H})$. Recall that the coadjoint action of $\mathfrak{b}^\pm_{1,2}(\mathcal{H})$ on its dual reads

$$\begin{array}{rcccl} -\mathrm{ad}^* & : & \mathfrak{b}^\pm_{1,2}(\mathcal{H}) \times \mathfrak{b}^\pm_{1,2}(\mathcal{H})^* & \longrightarrow & \mathfrak{b}^\pm_{1,2}(\mathcal{H})^* \\ & & (x,\alpha) & \longmapsto & -\mathrm{ad}^*_x\alpha := -\alpha \circ \mathrm{ad}_x. \end{array}$$

Let us show that $\mathfrak{u}_{\rm res}(\mathcal{H})$ is invariant under coadjoint action. This means that when α is given by $\alpha(y) = \Im\mathrm{Tr}\, ay$ for some $a \in \mathfrak{u}_{\rm res}(\mathcal{H})$, then, for any $x \in \mathfrak{b}^\pm_{1,2}(\mathcal{H})$, the one form $\beta = -\mathrm{ad}^*_x\alpha$ reads $\beta(y) = \Im\mathrm{Tr}\,\tilde{a}y$ for some $\tilde{a} \in \mathfrak{u}_{\rm res}(\mathcal{H})$. One has

$$\beta(y) = -\mathrm{ad}^*_x\alpha(y) = -\alpha(\mathrm{ad}_x y) = -\alpha([x,y]) = -\Im\mathrm{Tr}\, a[x,y] = -\Im\mathrm{Tr}\,(axy - ayx),$$

where $a \in \mathfrak{u}_{\rm res}(\mathcal{H})$, $x, y \in \mathfrak{b}^\pm_{1,2}(\mathcal{H})$. Since ay and x belong to $L^2(\mathcal{H})$, ayx and xay belong to $L^1(\mathcal{H})$ and $\mathrm{Tr}\,(ayx) = \mathrm{Tr}\,(xay)$. Since axy belongs also to $L^1(\mathcal{H})$, one has

$$\beta(y) = -\Im\mathrm{Tr}\,(axy) + \Im\mathrm{Tr}\,(ayx) = -\Im\mathrm{Tr}\,(axy) + \Im\mathrm{Tr}\,(xay) = -\Im\mathrm{Tr}\,([a,x]y).$$

Note that $[a,x]$ belongs to $L^2(\mathcal{H})$. Recall that by Proposition 3.4, the triples of Hilbert Lie algebras $(L^2(\mathcal{H}), \mathfrak{u}_2(\mathcal{H}), \mathfrak{b}^+_2(\mathcal{H}))$ and $(L^2(\mathcal{H}), \mathfrak{u}_2(\mathcal{H}), \mathfrak{b}^-_2(\mathcal{H}))$ are real Hilbert Manin triples with respect to the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{u}_2,\mathfrak{b}_2}$ given by the imaginary part of the trace. Using the decomposition $L^2(\mathcal{H}) = \mathfrak{u}_2(\mathcal{H}) \oplus \mathfrak{b}^+_2(\mathcal{H})$, and the continuous projection $p_{\mathfrak{u}_2^\pm} : L^2(\mathcal{H}) \to \mathfrak{u}_2(\mathcal{H})$ with kernal $\mathfrak{b}^\pm_2(\mathcal{H})$, one therefore has

$$\beta(y) = -\Im\mathrm{Tr}\, p_{\mathfrak{u}_2^\pm}([a,x])y,$$

since $y \in \mathfrak{b}^\pm_{1,2}(\mathcal{H}) \subset \mathfrak{b}^\pm_2(\mathcal{H})$ and $\mathfrak{b}^\pm_2(\mathcal{H})$ is isotropic. It follows that $\beta(y) = \Im\mathrm{Tr}\,\tilde{a}y$ with

$$\tilde{a} = -p_{\mathfrak{u}_2^\pm}([a,x]) \in \mathfrak{u}_2(\mathcal{H}) \subset \mathfrak{u}_{\rm res}(\mathcal{H}).$$

In other words, the coadjoint action of $x \in \mathfrak{b}_{1,2}^{\pm}(\mathcal{H})$ maps $a \in \mathfrak{u}_{\rm res}(\mathcal{H})$ to $-\mathrm{ad}^*_x a = -p_{\mathfrak{u}_2^{\pm}}([a,x]) \in \mathfrak{u}_{\rm res}(\mathcal{H})$. The continuity of the map

$$\begin{array}{rcccl} -\mathrm{ad}^* & : & \mathfrak{b}_{1,2}^{\pm}(\mathcal{H}) \times \mathfrak{u}_{\rm res}(\mathcal{H}) & \rightarrow & \mathfrak{u}_{\rm res}(\mathcal{H}) \\ & & (x,a) & \mapsto & -\mathrm{ad}^*_x a = -p_{\mathfrak{u}_2^{\pm}}([a,x]) \end{array}$$

follows from the continuity of the product

$$\begin{array}{ccl} \mathfrak{b}_{1,2}^{\pm}(\mathcal{H}) \times \mathfrak{u}_{\rm res}(\mathcal{H}) & \rightarrow & L^1(\mathcal{H}) \\ (x,a) & \mapsto & ax \end{array}$$

and from the continuity of $p_{\mathfrak{u}_2^{\pm}}$ and of the injections $L^1(\mathcal{H}) \subset L^2(\mathcal{H})$ and $\mathfrak{u}_2(\mathcal{H}) \subset \mathfrak{u}_{\rm res}(\mathcal{H})$.

Let us now prove (2). The dual map of the bilinear map $[\cdot,\cdot]_{\mathfrak{u}_{\rm res}}$ is given by

$$\begin{array}{rcccll} [\cdot,\cdot]^*_{\mathfrak{u}_{\rm res}} & : & \mathfrak{u}_{\rm res}(\mathcal{H})^* & \longrightarrow & L(\mathfrak{u}_{\rm res}(\mathcal{H}), \mathfrak{u}_{\rm res}(\mathcal{H}); \mathbb{K}) & \simeq L(\mathfrak{u}_{\rm res}(\mathcal{H}); \mathfrak{u}_{\rm res}(\mathcal{H})^*) \\ & & \mathcal{F}(\cdot) & \longmapsto & \mathcal{F}\left([\cdot,\cdot]_{\mathfrak{u}_{\rm res}}\right) & \mapsto \left(\alpha \mapsto \mathcal{F}\left([\alpha,\cdot]_{\mathfrak{u}_{\rm res}}\right) = \mathrm{ad}^*_{\alpha}\mathcal{F}(\cdot)\right), \end{array}$$

and takes values in $\Lambda^2\mathfrak{u}_{\rm res}(\mathcal{H})^*(\mathfrak{u}_{\rm res}(\mathcal{H}))$. Since by (1), $\mathfrak{u}_{\rm res}(\mathcal{H}) \subset \mathfrak{b}_{1,2}^{\pm}(\mathcal{H})^*$ is stable under the coadjoint action of $\mathfrak{b}_{1,2}^{\pm}(\mathcal{H})$ and the coadjoint action $\mathrm{ad}^* : \mathfrak{b}_{1,2}^{\pm}(\mathcal{H}) \times \mathfrak{u}_{\rm res}(\mathcal{H}) \rightarrow \mathfrak{u}_{\rm res}(\mathcal{H})$ is continuous, one can consider the adjoint action of $\mathfrak{b}_{1,2}^{\pm}(\mathcal{H})$ on $\Lambda^2\mathfrak{u}_{\rm res}(\mathcal{H})^*(\mathfrak{u}_{\rm res}(\mathcal{H}))$ defined by (4.3). Denote by θ the restriction of $[\cdot,\cdot]^*_{\mathfrak{u}_{\rm res}}$ to the subspace $\mathfrak{b}_{1,2}^{\pm}(\mathcal{H}) \subset \mathfrak{u}_{\rm res}(\mathcal{H})^*$:

$$\begin{array}{rcccll} \theta & : & \mathfrak{b}_{1,2}^{\pm}(\mathcal{H}) & \longrightarrow & L(\mathfrak{u}_{\rm res}(\mathcal{H}), \mathfrak{u}_{\rm res}(\mathcal{H}); \mathbb{K}) & \simeq L(\mathfrak{u}_{\rm res}(\mathcal{H}); \mathfrak{u}_{\rm res}(\mathcal{H})^*) \\ & & x & \longmapsto & \langle x, [\cdot,\cdot]_{\mathfrak{u}_{\rm res}(\mathcal{H})}\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} & \mapsto \big(\alpha \mapsto \langle x, [\alpha,\cdot]_{\mathfrak{u}_{\rm res}}\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} = \mathrm{ad}^*_{\alpha}x(\cdot)\big). \end{array}$$

The condition (5.4) that θ is a 1-cocycle reads :

$$\begin{array}{rl} \langle [x,y],[\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} & = +\langle y, [\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} + \langle y, [\alpha, \mathrm{ad}^*_x\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} \\ & \quad -\langle x, [\mathrm{ad}^*_y\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} - \langle x, [\alpha, \mathrm{ad}^*_y\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}}. \end{array} \tag{5.11}$$

The first term in the RHS reads

$$+\langle y, [\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} = \Im\mathrm{Tr}\, y[p_{\mathfrak{u}_2^{\pm}}([\alpha,x]),\beta] = \Im\mathrm{Tr}\,[\beta,y]p_{\mathfrak{u}_2^{\pm}}([\alpha,x]).$$

Using the fact that $[\beta,y] \in L^2(\mathcal{H})$, and that $\mathfrak{u}_2(\mathcal{H}) \subset L^2(\mathcal{H})$ and $\mathfrak{b}_2^{\pm}(\mathcal{H}) \subset L^2(\mathcal{H})$ are isotropic subspaces, one has

$$+\langle y, [\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} = \Im\mathrm{Tr}\, p_{\mathfrak{b}_2^{\pm}}([\beta,y])p_{\mathfrak{u}_2^{\pm}}([\alpha,x]).$$

Similarly the second, third and last term in the RHS of equation (5.11) read respectively

$$\begin{array}{rl} +\langle y, [\alpha,\mathrm{ad}^*_x\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} & = \Im\mathrm{Tr}\, p_{\mathfrak{b}_2^{\pm}}([y,\alpha])p_{\mathfrak{u}_2^{\pm}}([\beta,x]), \\ -\langle x, [\mathrm{ad}^*_y\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} & = -\Im\mathrm{Tr}\, p_{\mathfrak{b}_2^{\pm}}([\beta,x])p_{\mathfrak{u}_2^{\pm}}([\alpha,y]), \\ -\langle x, [\alpha,\mathrm{ad}^*_y\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} & = -\Im\mathrm{Tr}\, p_{\mathfrak{b}_2^{\pm}}([x,\alpha])p_{\mathfrak{u}_2^{\pm}}([\beta,y]). \end{array}$$

Using ones more the fact that $\mathfrak{u}_2(\mathcal{H}) \subset L^2(\mathcal{H})$ and $\mathfrak{b}_2^{\pm}(\mathcal{H}) \subset L^2(\mathcal{H})$ are isotropic subspaces, it follows that the first and last term in the RHS of equation (5.11) sum up to give

$$+\langle y, [\mathrm{ad}^*_x\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} - \langle x, [\alpha,\mathrm{ad}^*_y\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} = -\Im\mathrm{Tr}\,[\beta,y][x,\alpha],$$

and the second and third term in equation (5.11) simplify to

$$+\langle y, [\alpha,\mathrm{ad}^*_x\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} - \langle x, [\mathrm{ad}^*_y\alpha,\beta]\rangle_{\mathfrak{u}_{\rm res},\mathfrak{b}_{1,2}} = -\Im\mathrm{Tr}\,[\beta,x][\alpha,y].$$

Developping the brackets and using that for A and B bounded such that AB and BA are trace class one has $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$, the RHS of equation (5.11) becomes

$$\begin{aligned}\Im\mathrm{Tr}\,[\beta, y][x, \alpha] + \Im\mathrm{Tr}\,[\beta, x][\alpha, y] &= \Im\mathrm{Tr}\,(-\beta y x\alpha - y\beta\alpha x + \beta x y\alpha + x\beta\alpha y)\\ &= \Im\mathrm{Tr}\,(xy\alpha\beta - xy\beta\alpha - yx\alpha\beta + yx\beta\alpha)\\ &= \Im\mathrm{Tr}\,[x, y][\alpha, \beta]\\ &= \langle [x, y], [\alpha, \beta]\rangle_{\mathfrak{u}_{\mathrm{res}}, \mathfrak{b}_{1,2}}.\end{aligned}$$

One can show in a similar way that the Lie algebra structure $[\cdot,\cdot]_{\mathfrak{b}^{\pm}_{\mathrm{res}}}$ on $\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})$ is such that

(1) $\mathfrak{u}_{1,2}(\mathcal{H})$ acts continuously by coadjoint action on $\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})$;
(2) the dual map $[\cdot,\cdot]^*_{\mathfrak{b}^{\pm}_{\mathrm{res}}} : \mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})^* \to L(\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H}), \mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H}); \mathbb{K})$ to the Lie bracket $[\cdot,\cdot]_{\mathfrak{b}^{\pm}_{\mathrm{res}}} : \mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H}) \times \mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H}) \to \mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})$ restricts to a 1-cocycle $\theta : \mathfrak{u}_{1,2}(\mathcal{H}) \to \Lambda^2\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})^*(\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H}))$ with respect to the adjoint representation $\mathrm{ad}^{(2,0)}$ of $\mathfrak{u}_{1,2}(\mathcal{H})$ on $\Lambda^2\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})^*(\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H}))$.

□

5.6. **When there is no Manin triple where we expect one.** Now we will prove that the Banach space $\mathfrak{u}_{1,2}(\mathcal{H})$ is not a Banach Lie-Poisson spaces with respect to $\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})$. To prove this result, we will use the fact that the triangular truncation is unbounded on the space of trace class operators (cf Proposition 2.1). We construct in Example 5.22 a sequence of brackets $[x_n, y]$ between elements $x_n \in \mathfrak{u}_{1,2}(\mathcal{H})$ and an element $y \in \mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})$ such that the Hilbert-Schmidt norm of the diagonal blocks of $T_+[x_n, y]$ diverges. This allows to conclude in Lemma 5.24 that the coadjoint action of $\mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H})$ on $\mathfrak{u}_{1,2}(\mathcal{H})$ is unbounded. In a similar way, the coadjoint action of $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$ on $\mathfrak{b}^{+}_{1,2}(\mathcal{H})$ is unbounded. Using Theorem 5.16, we conclude that there is no natural Manin triple that can be built out of the pair $(\mathfrak{u}_{1,2}(\mathcal{H}), \mathfrak{b}^{\pm}_{\mathrm{res}}(\mathcal{H}))$, nor of the pair $(\mathfrak{b}^{+}_{1,2}(\mathcal{H}), \mathfrak{u}_{\mathrm{res}}(\mathcal{H}))$ (see Theorem 5.25 below).

Examples 5.22. Consider the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, with orthonormal basis $\{|n\rangle, n \in \mathbb{Z}\}$ ordered with respect to decreasing values of n, where $\mathcal{H}_+ = \mathrm{span}\{|n\rangle, n > 0\}$ and $\mathcal{H}_- = \mathrm{span}\{|n\rangle, n \leq 0\}$. Furthermore decompose $\mathcal{H}_+$ into the Hilbert sum of $\mathcal{H}_+^{\mathrm{even}} := \mathrm{span}\{|2n+2\rangle, n \in \mathbb{N}\}$ and $\mathcal{H}_+^{\mathrm{odd}} := \mathrm{span}\{|2n+1\rangle, n \in \mathbb{N}\}$. We will denote by $u : \mathcal{H}_+^{\mathrm{odd}} \to \mathcal{H}_+^{\mathrm{even}}$ the unitary operator defined by $u|2n+1\rangle = |2n+2\rangle$.

Since the triangular truncation is not bounded on the Banach space of trace class operators, there exists a sequence $K_n \in L^1(\mathcal{H}_+^{\mathrm{odd}})$ such that $\|K_n\|_1 \leq 1$ and $\|T_+(K_n)\|_1 > n$ for all $n \in \mathbb{N}$. It follows that either $\|T_+(K_n + K_n^*)/2\|_1 > n/2$ or $\|T_+(K_n - K_n^*)/2\|_1 > n/2$. Modulo the extraction of a subsequence, we can suppose that K_n is either hermitian $K_n = K_n^*$ or skew-hermitian $K_n = -K_n^*$. Moreover, since the triangular truncation is complex linear, the existence of a sequence of skew-hermitian operators such that $\|K_n\|_1 \leq 1$ and $\|T_+(K_n)\|_1 > n/2$ implies that the sequence iK_n is a sequence of hermitian operators such that $\|iK_n\|_1 \leq 1$ and $\|T_+(iK_n)\|_1 > n/2$. Therefore without loss of generality we can suppose that K_n are hermitian.

Consider the bounded operators x_n defined by 0 on $\mathcal{H}_-$, preserving $\mathcal{H}_+$ and whose expression with respect to the decomposition $\mathcal{H}_+ = \mathcal{H}_+^{\mathrm{even}} \oplus \mathcal{H}_+^{\mathrm{odd}}$ reads

$$x_n|_{\mathcal{H}_+} = \begin{pmatrix} 0 & uK_n \\ -K_n^* u^* & 0 \end{pmatrix}. \tag{5.12}$$

By construction, x_n is skew-hermitian. The restriction of $x_n^*x_n$ to $\mathcal{H}_+$ decomposes as follows with respect to $\mathcal{H}_+ = \mathcal{H}_+^{\text{even}} \oplus \mathcal{H}_+^{\text{odd}}$,

$$x_n^* x_n|_{\mathcal{H}_+} = \left(\begin{array}{cc} uK_n^*K_nu^* & 0 \\ 0 & K_n^*K_n \end{array}\right),$$

therefore x_n belongs to $\mathfrak{u}_{1,2}(\mathcal{H})$ since that singular values of x_n are the singular values of K_n but with doubled multiplicities. Moreover $\|x_n\|_1 \leq 2$.

Now let $y\,:\mathcal{H} \rightarrow \mathcal{H}$ be the bounded linear operator defined by 0 on $\mathcal{H}_+^{\text{even}}$, by 0 on $\mathcal{H}_-$, and by $y = u$ on $\mathcal{H}_+^{\text{odd}}$. Remark that y belongs to $\mathfrak{b}_{\text{res}}^+(\mathcal{H})$. Since x_n and y vanish on $\mathcal{H}_-$, one has

$$[x_n, y] = \left(\begin{array}{cc} [x_n, y]|_{\mathcal{H}_+} & 0 \\ 0 & 0 \end{array}\right),$$

where the operators $[x_n, y]|_{\mathcal{H}_+}$ have the following expression with respect to the decomposition $\mathcal{H}_+ = \mathcal{H}_+^{\text{even}} \oplus \mathcal{H}_+^{\text{odd}}$,

$$[x_n, y]|_{\mathcal{H}_+} = \left(\begin{array}{cc} uK_n^*u^* & 0 \\ 0 & -K_n^* \end{array}\right).$$

It follows that $\|T_+([x_n, y]|_{\mathcal{H}_+})\|_1 \rightarrow +\infty$.

Lemma 5.23. *Let $x_n \in \mathfrak{u}_{1,2}(\mathcal{H})$ and $y \in \mathfrak{b}_{\text{res}}^+(\mathcal{H})$ be as in example 5.22. Then $\|x_n\|_{\mathfrak{u}_{1,2}} \leq 2$ but $\|\text{ad}_y^* x_n\|_{\mathfrak{u}_{1,2}} \rightarrow +\infty$. In other words, the coadjoint action of $\mathfrak{b}_{\text{res}}^+(\mathcal{H})$ on $\mathfrak{u}_{1,2}(\mathcal{H})$ is unbounded.*

Proof. Consider the linear forms α_n on $\mathfrak{b}_{\text{res}}^+(\mathcal{H})$ given by $\alpha_n(A) = \Im \text{Tr}\, x_n A$ for $x_n \in \mathfrak{u}_{1,2}(\mathcal{H})$ defined by (5.12). Then the linear forms $\beta_n = -\text{ad}_y^* \alpha_n$ read

$$\beta_n(A) = -\text{ad}_y^*\alpha_n(A) = -\alpha_n(\text{ad}_y A) = -\alpha_n([y, A]) = -\Im \text{Tr}\, x[y, A] = -\Im \text{Tr}\,(xyA - xAy).$$

According to Proposition 2.1 in [GO10], one has $\text{Tr}\, xAy = \text{Tr}\, yxA$, therefore

$$\beta_n(A) = -\Im \text{Tr}\,[x_n, y]A.$$

The unique skew-symmetric operator T_n such that $-\Im \text{Tr}\, T_n A = -\Im \text{Tr}\,[x_n, y]A$ for any A in the subspace $\mathfrak{b}_2^+(\mathcal{H})$ of $\mathfrak{b}_{\text{res}}^+(\mathcal{H})$ is

$$T_n \quad = p_{\mathfrak{u}_2^+}([x_n, y]) = T_{--}([x_n, y]) - T_{--}([x_n, y])^* + \tfrac{1}{2}\left(D([x_n, y]) - D([x_n, y])^*\right)$$

Since K_n are hermitian, $[x_n, y]|_{\mathcal{H}_+}$ are hermitian and we get

$$T_n = [x_n, y] - 2T_+([x_n, y]) + D([x_n, y]).$$

In particular,

$$\|T_n\|_{\mathfrak{u}_{1,2}} + \|[x_n, y]\|_{\mathfrak{u}_{1,2}} + \|D([x_n, y])\|_{\mathfrak{u}_{1,2}} \geq \|T - [x_n, y] - D([x_n, y])\|_{\mathfrak{u}_{1,2}} \geq n,$$

for all $n \in \mathbb{N}$, and

$$\|T_n\|_{\mathfrak{u}_{1,2}} \geq n - 2 - \|D([x_n, y])\|_{\mathfrak{u}_{1,2}}.$$

The operator D consisting in taking the diagonal is bounded in $L^1(\mathcal{H})$ with operator norm less than 1, (see Theorem 1.19 in [Sim79] or [GK70] page 134) therefore

$$\|T_n\|_{\mathfrak{u}_{1,2}} > n - 4.$$

It follows that $\|-\text{ad}_y^*\alpha_n\|_{\mathfrak{u}_{1,2}} = \|T_n\|_{\mathfrak{u}_{1,2}} \rightarrow +\infty$.

□

Using the same kind of arguments, we have :

Lemma 5.24. *The coadjoint action of* $\mathfrak{u}_{\text{res}}(\mathcal{H})$ *on* $\mathfrak{b}^+_{1,2}(\mathcal{H})$ *is unbounded.*

It follows from the previous discussion , that we have the following Theorems :

Theorem 5.25. *The Banach Lie algebra* $\mathfrak{u}_{1,2}(\mathcal{H})$ *is not a Banach Lie-Poisson space with respect to* $\mathfrak{b}^\pm_{\text{res}}(\mathcal{H})$*. Consequently there is no natural Manin triple structure on the triple of Banach spaces* $(\mathfrak{b}^\pm_{\text{res}}(\mathcal{H}) \oplus \mathfrak{u}_{1,2}(\mathcal{H}), \mathfrak{b}^\pm_{\text{res}}(\mathcal{H}), \mathfrak{u}_{1,2}(\mathcal{H}))$*.*

Proof. The Banach space $\mathfrak{u}_{1,2}(\mathcal{H})$ are not Banach Lie-Poisson spaces with respect to $\mathfrak{b}^\pm_{\text{res}}(\mathcal{H})$ as a consequence of Lemma 5.24. By Theorem 5.16, there is no natural Manin triple structure on the Banach spaces $(\mathfrak{u}_{1,2}(\mathcal{H}) \oplus \mathfrak{b}^\pm_{\text{res}}(\mathcal{H}), \mathfrak{u}_{1,2}(\mathcal{H}), \mathfrak{b}^\pm_{\text{res}}(\mathcal{H}))$. □

Along the same lines, we have the analoguous Theorem :

Theorem 5.26. *The Banach Lie algebras* $\mathfrak{b}^\pm_{1,2}(\mathcal{H})$ *are not Banach Lie-Poisson spaces with respect to* $\mathfrak{u}_{\text{res}}(\mathcal{H})$*. Consequently there is no natural Manin triple structure on the triple of Banach spaces* $\left(\mathfrak{b}^\pm_{1,2}(\mathcal{H}) \oplus \mathfrak{u}_{\text{res}}(\mathcal{H}), \mathfrak{b}^\pm_{1,2}(\mathcal{H}), \mathfrak{u}_{\text{res}}(\mathcal{H})\right)$*.*

6. Example of Banach Poisson-Lie groups related to the restricted Grassmannian

The goal of this section is not to make a systematic theory of Banach Poisson Lie groups, but instead to define the examples of Poisson Lie groups that we will need in the following Sections.

6.1. **Definition of Banach Poisson-Lie groups.**

In order to be able to define the notion of Banach Poisson-Lie groups, we need to recall the construction of a Poisson structure on the product of two Poisson manifolds. The following Proposition is straightforward.

Proposition 6.1. *Let* $(M_1, \mathbb{F}_1, \pi_1)$ *and* $(M_2, \mathbb{F}_2, \pi_2)$ *be two Banach Poisson manifolds. Then the product* $M_1 \times M_2$ *carries a natural Banach Poisson manifold structure* $(M_1 \times M_2, \mathbb{F}, \pi)$ *where*

1. $M_1 \times M_2$ *carries the product Banach manifold structure, in particular the tangent bundle of* $M_1 \times M_2$ *is isomorphic to the direct sum* $TM_1 \oplus TM_2$ *of the vector bundles* TM_1 *and* TM_2 *and the cotangent bundle of* $M_1 \times M_2$ *is isomorphic to* $T^*M_1 \oplus T^*M_2$*,*
2. $\mathbb{F}$ *is the subbundle of* $T^*M_1 \oplus T^*M_2$ *defined as*
$$\mathbb{F}_{(p,q)} = (\mathbb{F}_1)_p \oplus (\mathbb{F}_2)_q,$$
3. π *is defined on* $\mathbb{F}$ *by*
$$\pi(\alpha_1 + \alpha_2, \beta_1 + \beta_2) = \pi_1(\alpha_1, \beta_1) + \pi_2(\alpha_2, \beta_2),$$
where $\alpha_1, \beta_1 \in \mathbb{F}_1$ *and* $\alpha_2, \beta_2 \in \mathbb{F}_2$*.*

Definition 6.2. Let $(M_1, \mathbb{F}_1, \pi_1)$ and $(M_2, \mathbb{F}_2, \pi_2)$ be two Banach Poisson manifolds and $F : M_1 \to M_2$ a smooth map. One say that F is a **Poisson map** at $p \in M_1$ if

1. the tangent map $T_pF : T_pM_1 \to T_{F(p)}M_2$ satisfies $T_pF^*(\mathbb{F}_2)_{F(p)} \subset (\mathbb{F}_1)_p$, i.e. for any covector $\alpha \in (\mathbb{F}_2)_{F(p)}$, the covector $\alpha \circ T_pF$ belongs to $(\mathbb{F}_1)_p$;
2. $(\pi_1)_p (\alpha \circ T_pF, \beta \circ T_pF) = (\pi_2)_{F(p)} (\alpha, \beta)$ for any $\alpha, \beta \in (\mathbb{F}_2)_{F(p)}$.

One says that F is a Poisson map if it is a Poisson map at any $p \in M_1$.

Definition 6.3. A **Banach Poisson-Lie group** G_+ is a Banach Lie group equipped with a Banach Poisson manifold structure such that the group multiplication $m : G_+ \times G_+ \to G_+$ is a Poisson map, where $G_+ \times G_+$ is endowed with the product Poisson structure.

Proposition 6.4. *A Banach Lie group G_+ equipped with a Banach Poisson structure $(G_+, \mathbb{F}, \pi)$ is a Banach Poisson-Lie group if and only if*

(1) $\mathbb{F}$ is invariant under left and right multiplications by elements in G_+,
(2) the Poisson tensor $\pi \in \Lambda^2\mathbb{F}^*(\mathbb{F})$ satisfies

$$\pi_{gu} = L_g^{**}\pi_u + R_u^{**}\pi_g, \quad \forall g, u \in G_+, \tag{6.1}$$

where L_g^{**} and R_u^{**} acts on π by

$$L_g^{**}\pi_u(\alpha, \beta) = \pi_u(L_g^*\alpha, L_g^*\beta)$$

and

$$R_g^{**}\pi_u(\alpha, \beta) = \pi_u(R_g^*\alpha, R_g^*\beta).$$

Proof. Let

$$\begin{array}{rcccl} m & : & G_+ \times G_+ & \to & G_+, \\ & & (g, u) & \mapsto & gu \end{array},$$

denote the multiplication in G_+.

(1) The tangent map $T_{(g,u)}m : T_gG_+ \oplus T_uG_+ \to T_{gu}G_+$ maps (X_g, X_u) to $T_gR_u(X_g) + T_uL_g(X_u)$. The first condition in Proposition 6.4 means that for any $\alpha \in \mathbb{F}_{gu}$, the covector $\alpha \circ T_uL_g$ belongs to $\mathbb{F}_u \subset T_u^*G_+$ and the covector $\alpha \circ T_gR_u$ belongs to $\mathbb{F}_g \subset T_g^*G_+$. This is equivalent to the first condition in definition 6.2.

(2) The multiplication m is a Poisson map if and only if

$$\pi_{G_+\times G_+}\left(\alpha \circ T_{(g,u)}m, \beta \circ T_{(g,u)}m\right) = \pi_{gu}(\alpha, \beta),$$

for any α and β in $\mathbb{F}_{gu}$. By definition of the Poisson structure on the product manifold $G_+ \times G_+$, one has :

$$\pi_{G_+\times G_+}\left(\alpha \circ T_{(g,u)}m, \beta \circ T_{(g,u)}m\right) = \pi_u\left(\alpha \circ T_uL_g, \beta \circ T_uL_g\right) + \pi_g\left(\alpha \circ T_gR_u, \beta \circ T_gR_u\right),$$

hence m a Poisson map if and only if (6.1) is satisfied.

□

Let us denote by $\mathrm{Ad} = L_g \circ R_g^{-1}$ the smooth adjoint action of a Lie group G_+ on its Lie algebra $\mathfrak{g}_+$, and by $\mathrm{Ad}^* = L_g^* \circ R_{g^{-1}}^*$ the induced smooth coadjoint action of G_+ on the dual space $\mathfrak{g}_+^*$. For any subspace $\mathfrak{g}_- \subset \mathfrak{g}_+^*$ invariant under the coadjoint action of G_+, the restriction

$$\begin{array}{rcccl} \mathrm{Ad}^* & : & G \times \mathfrak{g}_- & \to & \mathfrak{g}_- \\ & & (g, \beta) & \mapsto & \mathrm{Ad}^*(g)\beta, \end{array}$$

is continuous when $\mathfrak{g}_-$ is endowed with the norm of $\mathfrak{g}_+^*$, and one can define the coadjoint representation Ad^{**} of G_+ in $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$ by

$$\begin{array}{rcccl} \mathrm{Ad}^{**} & : & G & \longrightarrow & GL(\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)) \\ & & g & \longmapsto & \mathbf{t}(\cdot,\cdot) \mapsto \mathrm{Ad}^{**}(g)\mathbf{t} := \mathbf{t}(\mathrm{Ad}(g)^*\cdot, \mathrm{Ad}(g)^*\cdot). \end{array}$$

Theorem 6.5. *Let G_+ be a Banach Lie group and $(G_+, \mathbb{F}, \pi)$ a Banach Poisson structure on G_+. Then G_+ is a Banach Poisson-Lie group if and only if*

(1) $\mathbb{F}$ is invariant under left and right multiplications by elements in G_+,

(2) the subspace $\mathfrak{g}_- := \mathbb{F}_e$, where e is the unit element of G_+, is invariant under the coadjoint action of G_+ on $\mathfrak{g}_+^*$ and the map

$$\begin{array}{rcccl}\pi_r & : & G_+ & \rightarrow & \Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-) \\ & & g & \mapsto & R^{**}_{g^{-1}}\pi_g,\end{array}$$

is a 1-cocycle on G_+ with respect to the coadjoint representation of G_+ in $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$.

Proof. Since $\mathbb{F}$ is invariant under left and right multiplication, one has $\operatorname{Ad}^*(g)\mathbb{F}_e \subset \mathbb{F}_e$, for any $g \in G_+$. Using the relation $R^{**}_{(gu)^{-1}} = R^{**}_{g^{-1}} \circ R^{**}_{u^{-1}}$, the condition $\pi_{gu} = L^{**}_g\pi_u + R^{**}_u\pi_g$ for all $g, u \in G$ is equivalent to

$$R^{**}_{(gu)^{-1}}\pi_{gu} = R^{**}_{g^{-1}} \circ R^{**}_{u^{-1}} \circ L^{**}_g\pi_u + R^{**}_{g^{-1}} \circ R^{**}_{u^{-1}} \circ R^{**}_u\pi_g.$$

Since $R^{**}_{u^{-1}}$ and L^{**}_g commutes, the previous equality simplifies to give

$$\pi_r(gu) = R^{**}_{g^{-1}} \circ L^{**}_g\pi_r(u) + \pi_r(g) = \operatorname{Ad}(g)^{**}\pi_r(u) + \pi_r(g),$$

which is the cocycle condition (see Section 4.5). □

Remark 6.6. Let $(G_+, \mathbb{F}, \pi)$ be a Banach Poisson-Lie group and set $\mathfrak{g}_- := \mathbb{F}_e$. Denote by

$$\begin{array}{rcccl}\pi_r & : & G_+ & \rightarrow & \Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-) \\ & & g & \mapsto & R^{**}_{g^{-1}}\pi_g,\end{array}$$

the corresponding 1-cocycle on G_+ with respect to the coadjoint representation of G_+ in $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$. The differential of π_r at the unit element e of G_+ is a 1-cocycle $d\pi_r(e)\ : \mathfrak{g}_+ \rightarrow \Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$ with respect to the adjoint representation of $\mathfrak{g}_+$ on $\Lambda^2\mathfrak{g}_-^*(\mathfrak{g}_-)$. In the finite dimensional setting, the dual map of $d\pi_r(e)$ defines a Lie bracket on $\mathfrak{g}_-$. It is important to note that in the infinite-dimensional setting, the existence of a Lie bracket on $\mathfrak{g}_-$ is not guarantied by the existence of a Poisson Lie group structure on G_+.

6.2. **The dual Hilbert Poisson-Lie groups $\mathrm{U}_2(\mathcal{H})$ and $\mathrm{B}_2^{\pm}(\mathcal{H})$.** The proof of the following Theorem is left to the reader, but may be deduced from the proofs of Theorem 6.13 and Theorem 6.11.

Theorem 6.7. *Consider the Hilbert Lie group $\mathrm{U}_2(\mathcal{H})$ with Banach Lie algebras $\mathfrak{u}_2(\mathcal{H})$ and identify $\mathfrak{u}_2(\mathcal{H})^*$ with $\mathfrak{b}_2^{\pm}(\mathcal{H})$ via the application $b \mapsto (x \mapsto \Im\mathrm{Tr}\,(bx))$. Consider*

(1) $\pi_r^{\pm}\ : \mathrm{U}_2(\mathcal{H}) \rightarrow \Lambda^2\mathfrak{b}_2^{\pm}(\mathcal{H})^*(\mathfrak{b}_2^{\pm}(\mathcal{H}))$ defined by

$$\pi_r^{\pm}(u)(b_1, b_2) = \Im\mathrm{Tr}\,p_{\mathfrak{u}_2^{\pm}}(u^{-1}b_1u)\left[p_{\mathfrak{b}_2^{\pm}}(u^{-1}b_2u)\right],$$

(2) $\pi_g^{\pm} := R_g^{**}\pi_r^{\pm}(g)$.

Then $(\mathrm{U}_2(\mathcal{H}), T^*\,\mathrm{U}_2(\mathcal{H}), \pi^{\pm})$ is a Hilbert Poisson-Lie group.

Similarly, consider the Hilbert Lie group $\mathrm{B}_2^{\pm}(\mathcal{H})$ with Banach Lie algebra $\mathfrak{b}_2^{\pm}(\mathcal{H})$, and identify $\mathfrak{b}_2^{\pm}(\mathcal{H})^*$ with $\mathfrak{u}_2(\mathcal{H})$. Consider

(1) $\tilde{\pi}_r^{\pm}\ : \mathrm{B}_2^{\pm}(\mathcal{H}) \rightarrow \Lambda^2\mathfrak{u}_2(\mathcal{H})^*(\mathfrak{u}_2(\mathcal{H}))$ defined by

$$\tilde{\pi}_r^{\pm}(b)(x_1, x_2) = \Im\mathrm{Tr}\,p_{\mathfrak{b}_2^{\pm}}(b^{-1}x_1b)\left[p_{\mathfrak{u}_2^{\pm}}(b^{-1}x_2b)\right].$$

(2) $\tilde{\pi}_b^{\pm} := R_b^{**}\tilde{\pi}_r^{\pm}(b)$.

Then $(\mathrm{B}_2^{\pm}(\mathcal{H}), T^*\,\mathrm{B}_2^{\pm}(\mathcal{H}), \tilde{\pi}^{\pm})$ is a Hilbert Poisson-Lie group.

Proposition 6.8. *The derivatives of $\pi_r^{\pm}$ and $\tilde{\pi}_r^{\pm}$ at the unit element define bialgebra structures on $\mathfrak{u}_2(\mathcal{H})$ and $\mathfrak{b}_2^{\pm}(\mathcal{H})$ respectively, which combine into the Manin triple $(L^2(\mathcal{H}), \mathfrak{u}_2(\mathcal{H}), \mathfrak{b}_2^{\pm}(\mathcal{H}))$ given in Proposition 3.4.*

Proof. The derivative of $\pi_r^{\pm}$ at the unit element of $\mathrm{U}_2(\mathcal{H})$ reads :

$$\begin{array}{rcl} d\pi_r^{\pm}(e)(x)(b_1, b_2) = & \Im\mathrm{Tr}\left(p_{\mathfrak{u}_2^{\pm}}([x, b_1])p_{\mathfrak{b}_2^{\pm}}(b_2)\right) + \Im\mathrm{Tr}\left(p_{\mathfrak{u}_2^{\pm}}(b_1)p_{\mathfrak{b}_2^{\pm}}([x, b_2])\right), \\ = & \Im\mathrm{Tr}\left(p_{\mathfrak{u}_2^{\pm}}([x, b_1])b_2\right) = \Im\mathrm{Tr}\,[x, b_1]b_2 = \Im\mathrm{Tr}\,x[b_1, b_2]_{\mathfrak{b}_2^{\pm}}, \end{array}$$

where we have use that $\mathfrak{b}_2^{\pm}(\mathcal{H})$ is an isotropic subspace. Similarly,

$$d\tilde{\pi}_r^{\pm}(e)(b)(x_1, x_2) = \Im\mathrm{Tr}\,b[x_1, x_2]_{\mathfrak{u}_2},$$

which is the dual map of the bracket $[\cdot,\cdot]_{\mathfrak{u}_2}$. □

6.3. **The Banach Poisson-Lie groups $\mathrm{B}_{\mathrm{res}}^{\pm}(\mathcal{H})$ and $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$.** In this Section we will construct a Banach Poisson-Lie group structure on the Banach Lie group $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$. A similar construction can be of course carry out for the Banach Lie group $\mathrm{B}_{\mathrm{res}}^{-}(\mathcal{H})$ instead. Recall that the coajoint action of $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$ is unbounded on $\mathfrak{u}_{1,2}(\mathcal{H})$ (see Section 5.6, in particular Lemma 5.24). Therefore, in order to construct a Poisson-Lie group structure on $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$, we need a larger subspace of the dual $\mathfrak{b}_{\mathrm{res}}^{+}(\mathcal{H})^*$ which will play the role of $\mathfrak{g}_- := \mathbb{F}_e$ (compare with Theorem 6.5). Consider the following application :

$$\begin{array}{rcccl} F & : & L_{1,2}(\mathcal{H}) & \to & \mathfrak{b}_{\mathrm{res}}^{+}(\mathcal{H})^* \\ & & a & \mapsto & (b \mapsto \Im\mathrm{Tr}\,ab)\,. \end{array}$$

Proposition 6.9. *The kernel of F equals $\mathfrak{b}_{1,2}^{+}(\mathcal{H})$, therefore $L_{1,2}(\mathcal{H})/\mathfrak{b}_{1,2}^{+}(\mathcal{H})$ injects into the dual space $\mathfrak{b}_{\mathrm{res}}^{+}(\mathcal{H})^*$. Moreover $L_{1,2}(\mathcal{H})/\mathfrak{b}_{1,2}^{+}(\mathcal{H})$ is preserved by the coadjoint action of $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$ and strictly contains $\mathfrak{u}_{1,2}(\mathcal{H})$ as a dense subspace.*

Proof. In order to show that the kernel of F is $\mathfrak{b}_{1,2}^{+}(\mathcal{H})$, consider, for $n \geq m$, the operator $E_{nm} \in \mathfrak{b}_{\mathrm{res}}^{+}(\mathcal{H})$ given by $x \mapsto \langle x, m\rangle |n\rangle$ and the operator $iE_{nm} \in \mathfrak{b}_{\mathrm{res}}^{+}(\mathcal{H})$. As in the proof of Proposition 5.19, an element a satisfying $F(a)(E_{nm}) = 0$ and $F(a)(iE_{nm}) = 0$ is such that $\langle m, an\rangle = 0$ for $n > m$ and $\langle n, an\rangle \in \mathbb{R}$ for $n \in \mathbb{Z}$, i.e. belongs to $\mathfrak{b}_{1,2}^{+}(\mathcal{H})$. Let us show that the range of F is preserved by the coadjoint action of $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$. Let $g \in \mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$ and $a \in L_{1,2}(\mathcal{H})$. For any $b \in \mathfrak{b}_{\mathrm{res}}^{+}(\mathcal{H})$, one has :

$$\mathrm{Ad}^*(g)F(a)(b) = F(a)(\mathrm{Ad}(g)(b)) = F(a)(gbg^{-1}) = \Im\mathrm{Tr}\,agbg^{-1} = \Im\mathrm{Tr}\,g^{-1}agb = F(g^{-1}ag)(b),$$

where, in the forth equality, we have used Proposition 2.1 in [GO10] (since the product agb belongs to $L_{1,2}(\mathcal{H})$ and b to $L_{\mathrm{res}}(\mathcal{H})$). In fact, $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$ acts continuously on the right on $L_{1,2}(\mathcal{H})$ by

$$a \cdot g = g^{-1}ag.$$

Then one has the equivariance property

$$F(a \cdot g) = \mathrm{Ad}^*(g)F(a).$$

Moreover the subalgebra $\mathfrak{b}_{1,2}^{+}(\mathcal{H})$ is preserved by the right action of $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$ on $L_{1,2}(\mathcal{H})$. It follows that there is a well-defined right action of $\mathrm{B}_{\mathrm{res}}^{+}(\mathcal{H})$ on the quotient space $L_{1,2}(\mathcal{H})/\mathfrak{b}_{1,2}^{+}(\mathcal{H})$ defined by

$$[a] \cdot g = [a \cdot g],$$

where $[a]$ denotes the class of $a \in L_{1,2}(\mathcal{H})$ modulo $\mathfrak{b}_{1,2}^{+}(\mathcal{H})$.

Let us show that $\mathfrak{u}_{1,2}(\mathcal{H}) \oplus \mathfrak{b}^+_{1,2}(\mathcal{H})$ is dense in $L_{1,2}(\mathcal{H})$. To do this, we will show that any continuous linear form on $L_{1,2}(\mathcal{H})$ which vanishes on $\mathfrak{u}_{1,2}(\mathcal{H}) \oplus \mathfrak{b}^+_{1,2}(\mathcal{H})$ is equal to the zero form. Recall that the dual space of $L_{1,2}(\mathcal{H})$ is $L_{\rm res}(\mathcal{H})$, the duality pairing being given by the trace. Consider $X \in L_{\rm res}(\mathcal{H})$ such that $\operatorname{Tr} Xa = 0$ and $\operatorname{Tr} Xb = 0$ for any $a \in \mathfrak{u}_{1,2}(\mathcal{H})$ and any $b \in \mathfrak{b}^+_{1,2}(\mathcal{H})$. Letting $b = E_{nm}$ with $n \geq m$, we get $\langle m, Xn \rangle = 0$ for $n \geq m$. Letting $a = E_{nm} - E_{mn} \in \mathfrak{u}_{1,2}(\mathcal{H})$, we get $\langle m, Xn \rangle - \langle n, Xm \rangle = 0$ for $n \geq m$. It follows that $\langle m, Xn \rangle = 0$ for any $m, n \in \mathbb{Z}$, which implies that the bounded linear operator X vanishes. It follows from Section 5.6, that $\mathfrak{u}_{1,2}(\mathcal{H}) \oplus \mathfrak{b}^+_{1,2}(\mathcal{H})$ is strictly contained in $L_{1,2}(\mathcal{H})$.

Let us show that $\mathfrak{u}_{1,2}(\mathcal{H})$ is dense in $L_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H})$. Consider a class $[a] \in L_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H})$, where a is any element in $L_{1,2}(\mathcal{H})$. Since $\mathfrak{u}_{1,2}(\mathcal{H}) \oplus \mathfrak{b}^+_{1,2}(\mathcal{H})$ is dense in $L_{1,2}(\mathcal{H})$, there is a sequence $u_i \in \mathfrak{u}_{1,2}(\mathcal{H})$ and a sequence $b_i \in \mathfrak{b}^+_{1,2}(\mathcal{H})$ such that $u_i + b_i$ converge to a in $L_{1,2}(\mathcal{H})$. It follows that $[u_i + b_i] = [u_i]$ converge to $[a]$ in $L_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H})$. □

In order to verify the Jacobi identity (5.1) for a Poisson tensor on a Banach Lie group, we will need the following lemma :

Lemma 6.10. *Let* B *be a Banach Lie group with Lie algebra* $\mathfrak{b}$, $\mathbb{B}$ *a subbundle of* T^*B *in duality with* TB, *invariant by left and right translations of* B, *and* π *a smooth section of* $\Lambda^2\mathbb{B}^*(\mathbb{B})$.

1. *any closed local section* α *of* $\mathbb{B}$ *in a neighborhood* $\mathcal{V}_b$ *of* $b \in \mathrm{B}$ *is of the form* $\alpha = R_b^*\alpha_0$, *where* $\alpha_0 : \mathcal{V}_b \to \mathbb{B}_e \subset \mathfrak{b}^*$ *satisfies :*

$$\langle \alpha_0(b), [X_0, Y_0] \rangle = \langle T_b\alpha_0(R_bY_0), X_0 \rangle - \langle T_b\alpha_0(R_bX_0), Y_0 \rangle, \tag{6.2}$$

   with $T_b\alpha_0 : T_b\mathrm{B} \to \mathfrak{b}^*$ *the tangent map of* α_0 *at* $b \in \mathcal{V}_b$, *and* X_0, Y_0 *any elements in* $\mathfrak{b}$.

2. *Let* $\pi_r : \mathrm{B} \to \Lambda^2\mathbb{B}^*_e(\mathbb{B}_e)$ *be defined by* $\pi_r(b) := R^{**}_{b^{-1}}\pi$. *Then for any closed local sections* α, β *of* $\mathbb{B}$ *around* $b \in \mathrm{B}$, *the differential* $d(\pi(\alpha,\beta))$ *at* b *reads*

$$d(\pi(\alpha,\beta))(X_b) = T_b\pi_r(X_b)(\alpha_0(b),\beta_0(b)) + \pi_r(b)(T_b\alpha_0(X_b),\beta_0(b)) + \pi_r(b)(\alpha_0(b), T_b\beta_0(X_b)),$$

   where $X_b \in T_b\mathrm{B}$, $T_b\pi_r : T_b\mathrm{B} \to \Lambda^2\mathbb{B}^*_e(\mathbb{B}_e)$ *is the tangent map,* $\alpha = R_b^*\alpha_0$ *and* $\beta = R_b^*\beta_0$.

3. *Suppose that* $i_{\alpha_0}\pi_r(b) \in \mathfrak{b} \subset \mathbb{B}^*$ *for any* $\alpha \in \mathbb{B}$. *Then for any closed local sections* α, β, γ *of* $\mathbb{B}$,

$$\begin{aligned}
&\pi(\alpha, d(\pi(\beta,\gamma))) + \pi(\beta, d(\pi(\gamma,\alpha))) + \pi(\gamma, d(\pi(\alpha,\beta))) = \\
&\quad T_b\pi_r(R_b i_{\alpha_0}\pi_r(b))(\beta_0(b),\gamma_0(b)) + \langle \alpha_0(b), [i_{\gamma_0(b)}\pi_r(b), i_{\beta_0(b)}\pi_r(b)] \rangle \\
&\quad + T_b\pi_r(R_b i_{\beta_0}\pi_r(b))(\gamma_0(b),\alpha_0(b)) + \langle \beta_0(b), [i_{\alpha_0(b)}\pi_r(b), i_{\gamma_0(b)}\pi_r(b)] \rangle \\
&\quad + T_b\pi_r(R_b i_{\gamma_0}\pi_r(b))(\alpha_0(b),\beta_0(b)) + \langle \gamma_0(b), [i_{\beta_0(b)}\pi_r(b), i_{\alpha_0(b)}\pi_r(b)] \rangle
\end{aligned} \tag{6.3}$$

   where $\alpha = R_b^*\alpha_0$, $\beta = R_b^*\beta_0$, *and* $\gamma = R_b^*\gamma_0$. *In particular the left hand side of equation* (6.3) *defines a tensor.*

Proof. (1) Since α is closed, one has :

$$d\alpha_b(X,Y) = \mathcal{L}_X\alpha(Y) - \mathcal{L}_Y\alpha(X) - \alpha([X,Y]) = 0$$

for any local vector fields X and Y around $b \in \mathcal{V}_b$. But since $d\alpha$ is a tensor (see Proposition 3.2, chapter V in [La01]), the previous identity depends only on the values of X and Y at b. In other words, α is closed if and only if the previous identity is satisfied for any right invariant vector fields X and Y. Set $X_b = R_b X_0$ and $Y_b = R_b Y_0$ for $X_0, Y_0 \in \mathfrak{b}$. One has

$$\begin{aligned} d\alpha(X,Y) &= \mathcal{L}_X\alpha_0(b)(R_{b^{-1}}Y_b) - \mathcal{L}_Y\alpha_0(b)(R_{b^{-1}}X_b) - \alpha_0(b)(R_{b^{-1}}[X,Y]_b) \\ &= \mathcal{L}_X\alpha_0(b)(Y_0) - \mathcal{L}_Y\alpha_0(b)(X_0) + \alpha_0(b)([X_0,Y_0]_{\mathfrak{b}}) \end{aligned}$$

Denote by $f\ : \mathcal{V}_b \to \mathbb{R}$ the function defined by $f(b) = \alpha_0(b)(Y_0) = \langle \alpha_0(b), Y_0\rangle$, where the bracket denotes the natural pairing between $\mathfrak{b}^*$ and $\mathfrak{b}$. Then

$$df_b(X_b) = \langle T_b\alpha_0(R_bX_0), Y_0\rangle.$$

It follows that

$$d\alpha(X,Y) = \langle T_b\alpha_0(R_bX_0), Y_0\rangle - \langle T_b\alpha_0(R_bY_0), X_0\rangle + \langle \alpha_0(b), [X_0,Y_0]_{\mathfrak{b}}\rangle.$$

Therefore $d\alpha(X,Y) = 0$ for any X and Y if and only if

$$\langle \alpha_0(b), [X_0,Y_0]\rangle = \langle T_b\alpha_0(R_bY_0), X_0\rangle - \langle T_b\alpha_0(R_bX_0), Y_0\rangle,$$

for any X_0 and Y_0 in $\mathfrak{b}$.

(2) This is a straighforward application of the chain rule.

(3) In the case where $i_{\alpha_0}\pi_r(b)$ belongs to $\mathfrak{b}$, one has the following expression of the differential of π :

$$d\left(\pi(\beta,\gamma)\right)(X_b) = T_b\pi_r(X_b)(\beta_0(b),\gamma_0(b)) - \langle T_b\beta_0(X_b), i_{\gamma_0(b)}\pi_r(b)\rangle + \langle T_b\gamma_0(X_b), i_{\beta_0(b)}\pi_r(b)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathfrak{b}^*$ and $\mathfrak{b}$. Therefore

$$\begin{aligned} \pi(\alpha, d\left(\pi(\beta,\gamma)\right) &= \pi_r(b)\left(\alpha_0(b), R_b^* d\left(\pi(\beta,\gamma)\right)\right) = d\left(\pi(\beta,\gamma)\right)\left(R_b i_{\alpha_0(b)}\pi_r(b)\right) \\ &= T_b\pi_r(R_b i_{\alpha_0(b)}\pi_r(b))(\beta_0(b),\gamma_0(b)) \\ &\quad -\langle T_b\beta_0(R_b i_{\alpha_0(b)}\pi_r(b)), i_{\gamma_0(b)}\pi_r(b)\rangle \\ &\quad +\langle T_b\gamma_0(R_b i_{\alpha_0(b)}\pi_r(b)), i_{\beta_0(b)}\pi_r(b)\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} &\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) \\ &= T_b\pi_r(R_b i_{\alpha_0}\pi_r(b))\left(\beta_0(b),\gamma_0(b)\right) + T_b\pi_r(R_b i_{\beta_0}\pi_r(b))\left(\gamma_0(b),\alpha_0(b)\right) + T_b\pi_r(R_b i_{\gamma_0}\pi_r(b))\left(\alpha_0(b),\beta_0(b)\right) \\ &-\langle T_b\beta_0(R_b i_{\alpha_0(b)}\pi_r(b)), i_{\gamma_0(b)}\pi_r(b)\rangle + \langle T_b\gamma_0(R_b i_{\alpha_0(b)}\pi_r(b)), i_{\beta_0(b)}\pi_r(b)\rangle \\ &-\langle T_b\gamma_0(R_b i_{\beta_0(b)}\pi_r(b)), i_{\alpha_0(b)}\pi_r(b)\rangle + \langle T_b\alpha_0(R_b i_{\beta_0(b)}\pi_r(b)), i_{\gamma_0(b)}\pi_r(b)\rangle \\ &-\langle T_b\alpha_0(R_b i_{\gamma_0(b)}\pi_r(b)), i_{\beta_0(b)}\pi_r(b)\rangle + \langle T_b\beta_0(R_b i_{\gamma_0(b)}\pi_r(b)), i_{\alpha_0(b)}\pi_r(b)\rangle \end{aligned}$$

Using (6.2), the previous equation simplifies to

$$\begin{aligned} &\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) \\ &= T_b\pi_r(R_b i_{\alpha_0}\pi_r(b))\left(\beta_0(b),\gamma_0(b)\right) + T_b\pi_r(R_b i_{\beta_0}\pi_r(b))\left(\gamma_0(b),\alpha_0(b)\right) + T_b\pi_r(R_b i_{\gamma_0}\pi_r(b))\left(\alpha_0(b),\beta_0(b)\right) \\ &+\langle \alpha_0(b), [i_{\gamma_0(b)}\pi_r(b), i_{\beta_0(b)}\pi_r(b)]\rangle + \langle \beta_0(b), [i_{\alpha_0(b)}\pi_r(b), i_{\gamma_0(b)}\pi_r(b)]\rangle + \langle \gamma_0(b), [i_{\beta_0(b)}\pi_r(b), i_{\alpha_0(b)}\pi_r(b)]\rangle. \end{aligned}$$

□

Now we are able to state the following Theorem.

Theorem 6.11. *Consider the Banach Lie group* $\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$*, and*

(1) $\mathfrak{g}_- := \mathrm{L}_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H}) \subset \mathfrak{b}^+_{\mathrm{res}}(\mathcal{H})^*$*,*

(2) $\mathbb{B} \subset T^*\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$*,* $\mathbb{B}_b := R^*_{b^{-1}}\mathfrak{g}_-$*,*

(3) $\pi_r\ : \mathrm{B}^+_{\mathrm{res}}(\mathcal{H}) \to \Lambda^2\mathfrak{g}^*_-(\mathfrak{g}_-)$ *defined by*

$$\pi_r(b)([x_1]_{\mathfrak{b}^+_{1,2}}, [x_2]_{\mathfrak{b}^+_{1,2}}) = \Im\mathrm{Tr}\,(b^{-1}\,p_{\mathfrak{u}_2}(x_1)\,b)\left[p_{\mathfrak{b}^+_2}(b^{-1}\,p_{\mathfrak{u}_2}(x_2)\,b)\right],$$

(4) $\pi(b) = R^{**}_b\pi_r(b)$.

Then $\left(\mathrm{B}^+_{\mathrm{res}}(\mathcal{H}), \mathbb{B}, \pi\right)$ *is a Banach Poisson-Lie group.*

Proof.
- Let us show that π_r satisfies the cocycle condition.

$$\pi_r(u)\left(\mathrm{Ad}^*(g)[x_1]_{\mathfrak{b}^+_{1,2}}, \mathrm{Ad}^*(g)[x_2]_{\mathfrak{b}^+_{1,2}}\right) = \pi_r(u)\left([g^{-1}\,x_1\,g]_{\mathfrak{b}^+_{1,2}}, [g^{-1}x_2\,g]_{\mathfrak{b}^+_{1,2}}\right)$$
$$= \Im\mathrm{Tr}\,(u^{-1}p_{\mathfrak{u}^+_2}(g^{-1}x_1\,g)\,u)\left[p_{\mathfrak{b}^+_2}(u^{-1}p_{\mathfrak{u}^+_2}(g^{-1}x_2\,g)u)\right]$$

Using the decomposition $p_{\mathfrak{u}^+_2}(g^{-1}x_1\,g) = g^{-1}x_1\,g - p_{\mathfrak{b}^+_2}(g^{-1}x_1\,g)$, the fact that $\mathfrak{b}^+_2$ is preserved by conjugation by elements in $\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$, and the fact that $\mathfrak{b}^+_2$ is isotropic, one has :

$$\pi_r(u)\left(\mathrm{Ad}^*(g)[x_1]_{\mathfrak{b}^+_{1,2}}, \mathrm{Ad}^*(g)[x_2]_{\mathfrak{b}^+_{1,2}}\right) = \Im\mathrm{Tr}\,(u^{-1}g^{-1}x_1\,g\,u)\left[p_{\mathfrak{b}^+_2}(u^{-1}p_{\mathfrak{u}^+_2}(g^{-1}x_2\,g)u)\right]$$
$$= \Im\mathrm{Tr}\,(u^{-1}g^{-1}x_1\,g\,u)\left[p_{\mathfrak{b}^+_2}(u^{-1}g^{-1}x_2\,gu)\right] - \Im\mathrm{Tr}\,(u^{-1}g^{-1}x_1\,g\,u)\left[p_{\mathfrak{b}^+_2}(u^{-1}p_{\mathfrak{b}^+_2}(g^{-1}x_2\,g)u)\right]$$
$$= \Im\mathrm{Tr}\,(u^{-1}g^{-1}x_1\,g\,u)\left[p_{\mathfrak{b}^+_2}(u^{-1}g^{-1}x_2\,gu)\right] - \Im\mathrm{Tr}\,g^{-1}x_1\,g\,p_{\mathfrak{b}^+_2}(g^{-1}x_2\,g)$$

Using the decompositions $x_1 = p_{\mathfrak{u}^+_2}(x_1) + p_{\mathfrak{b}^+_2}(x_1)$ and $x_2 = p_{\mathfrak{u}^+_2}(x_2) + p_{\mathfrak{b}^+_2}(x_2)$, one gets 8 terms but 4 of them vanish since $\mathfrak{b}^+_2$ is isotropic. The remaining terms are:

$$\begin{aligned}\pi_r(u)\left(\mathrm{Ad}^*(g)[x_1]_{\mathfrak{b}^+_{1,2}}, \mathrm{Ad}^*(g)[x_2]_{\mathfrak{b}^+_{1,2}}\right) &= \Im\mathrm{Tr}\,(u^{-1}g^{-1}p_{\mathfrak{u}^+_2}(x_1)\,g\,u)\left[p_{\mathfrak{b}^+_2}(u^{-1}g^{-1}p_{\mathfrak{u}^+_2}(x_2)\,gu)\right]\\ &+\Im\mathrm{Tr}\,(u^{-1}g^{-1}p_{\mathfrak{u}^+_2}(x_1)\,g\,u)\left[p_{\mathfrak{b}^+_2}(u^{-1}g^{-1}p_{\mathfrak{b}^+_2}(x_2)\,gu)\right]\\ &-\Im\mathrm{Tr}\,g^{-1}p_{\mathfrak{u}^+_2}(x_1)\,g\,p_{\mathfrak{b}^+_2}(g^{-1}p_{\mathfrak{u}^+_2}(x_2)\,g)\\ &-\Im\mathrm{Tr}\,g^{-1}p_{\mathfrak{u}^+_2}(x_1)\,g\,p_{\mathfrak{b}^+_2}(g^{-1}p_{\mathfrak{b}^+_2}(x_2)\,g)\end{aligned}$$

The first term in the right hand side equals $\pi_r(gu)([x_1]_{\mathfrak{b}^+_{1,2}}, [x_2]_{\mathfrak{b}^+_{1,2}})$, the third term equals $-\pi_r(g)([x_1]_{\mathfrak{b}^+_{1,2}}, [x_2]_{\mathfrak{b}^+_{1,2}})$, whereas the second terms equals $+\Im\mathrm{Tr}\,(p_{\mathfrak{u}^+_2}(x_1)p_{\mathfrak{b}^+_2}(x_2))$, and the last terms equals $-\Im\mathrm{Tr}\,(p_{\mathfrak{u}^+_2}(x_1)p_{\mathfrak{b}^+_2}(x_2))$.
- It remains to check that π satisfies the Jacobi identity (5.1). Using the cocycle identity, one has for any X in $\mathfrak{b}^+_{\mathrm{res}}(\mathcal{H})$ and $g \in \mathrm{B}^+_{\mathrm{res}}$,

$$T_g\pi_r(L_gX)([x_1], [x_2]) = T_e\pi_r(X)(\mathrm{Ad}^*(g)[x_1], \mathrm{Ad}^*(g)[x_2]),$$

in particular,

$$\begin{aligned}T_g\pi_r(R_gX)([x_1], [x_2]) &= T_g\pi_r(L_g\mathrm{Ad}(g^{-1})(X))([x_1], [x_2])\\ &= T_e\pi_r(\mathrm{Ad}(g^{-1})(X))(\mathrm{Ad}^*(g)[x_1], \mathrm{Ad}^*(g)[x_2])\\ &= T_e\pi_r(\mathrm{Ad}(g^{-1})(X))([g^{-1}\,x_1\,g], [g^{-1}\,x_2\,g])\end{aligned}$$

On the other hand

$$\begin{aligned}T_e\pi_r(Y)([x_1], [x_2]) &= -\Im\mathrm{Tr}\,[Y, p_{\mathfrak{u}^+_2}(x_1)]p_{\mathfrak{b}^+_2}(p_{\mathfrak{u}^+_2}(x_2)) - \Im\mathrm{Tr}\,p_{\mathfrak{u}^+_2}(x_1)p_{\mathfrak{b}^+_2}([Y, p_{\mathfrak{u}^+_2}(x_2)])\\ &= -\Im\mathrm{Tr}\,p_{\mathfrak{u}^+_2}(x_1)[Y, p_{\mathfrak{u}^+_2}(x_2)] = \Im\mathrm{Tr}\,Y[p_{\mathfrak{u}^+_2}(x_1), p_{\mathfrak{u}^+_2}(x_2)].\end{aligned}$$

It follows that

$$T_g\pi_r(R_gX)([x_1], [x_2]) = \Im\mathrm{Tr}\,g^{-1}\,X\,g[p_{\mathfrak{u}^+_2}(g^{-1}\,x_1\,g), p_{\mathfrak{u}^+_2}(g^{-1}\,x_2\,g)]. \tag{6.4}$$

In particular, for any x_1 and x_2 in $\mathrm{L}_{1,2}(\mathcal{H})$, the 1-form on $\mathfrak{b}^+_{\mathrm{res}}$ given by

$$X \mapsto T_g\pi_r(L_gX)([x_1], [x_2])$$

belongs to $\mathfrak{u}_{1,2}(\mathcal{H})$ and is given by

$$T_g\pi_r(L_g(\cdot))([x_1], [x_2]) = [p_{\mathfrak{u}^+_2}(g^{-1}\,x_1\,g), p_{\mathfrak{u}^+_2}(g^{-1}\,x_2\,g)]$$

Moreover

$$\begin{aligned}\pi_r(g)([x_3],[\cdot]) &= \Im\mathrm{Tr}\,(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(\cdot)\,g)\\ &= \Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(\cdot)\,g)\\ &= -\Im\mathrm{Tr}\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(\cdot)\,g)\\ &= -\Im\mathrm{Tr}\,g\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)\,g^{-1}\,p_{\mathfrak{u}_2^+}(\cdot)\\ &= -\Im\mathrm{Tr}\,g\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)\,g^{-1}(\cdot)\end{aligned}$$

In particular $i_{[x_3]}\pi_r(g) = -g\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)\,g^{-1}$ belongs to $\mathfrak{b}_2^+(\mathcal{H}) \subset \mathfrak{b}_{\mathrm{res}}^+(\mathcal{H})$. Using (6.4), it follows that

(6.5)
$$\begin{aligned}T_g\pi_r(R_g i_{[x_3]}\pi_r(g))([x_1],[x_2]) &= -\Im\mathrm{Tr}\;p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}\,x_1\,g),p_{\mathfrak{u}_2^+}(g^{-1}\,x_2\,g)]\\ &= -\Im\mathrm{Tr}\;p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)],\end{aligned}$$

where we have used that $g^{-1}\,p_{b_2^+}(x_i)\,g \in \mathfrak{b}_2^+$ for any $x_i \in \mathrm{L}_{1,2}(\mathcal{H})$ and any $g \in \mathrm{B}_{\mathrm{res}}^+(\mathcal{H})$. Moreover

(6.6)
$$\begin{aligned}\langle x_1,[i_{[x_3]}\pi_r(g),i_{[x_2]}\pi_r(g)]\rangle &= \Im\mathrm{Tr}\,x_1[g\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)\,g^{-1},g\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)\,g^{-1}]\\ &= \Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(x_1)[g\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)\,g^{-1},g\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)\,g^{-1}]\\ &= \Im\mathrm{Tr}\,g^{-1}p_{\mathfrak{u}_2^+}(x_1)g[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ &= \Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_1)g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ &= -\Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_1)g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)]\end{aligned}$$

Consider $\alpha = R_{g^{-1}}^*[x_1] \in (T_g\,\mathrm{B}_{\mathrm{res}}^+)^*$, $\beta = R_{g^{-1}}^*[x_2] \in (T_g\,\mathrm{B}_{\mathrm{res}}^+)^*$ and $\gamma = R_{g^{-1}}^*[x_3] \in (T_g\,\mathrm{B}_{\mathrm{res}}^+)^*$, for x_1, x_2 and x_3 in $\mathrm{L}_{1,2}(\mathcal{H})$. Injecting (6.5) and (6.6) into (6.3) and using the fact that the left hand side of (6.3) defines a tensor, one gets :

$$\begin{aligned}&\pi(\alpha,d(\pi(\beta,\gamma)))+\pi(\beta,d(\pi(\gamma,\alpha)))+\pi(\gamma,d(\pi(\alpha,\beta)))\\ &\quad= -\Im\mathrm{Tr}\;p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ &\qquad-\Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_1)g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)]\\ &\qquad-\Im\mathrm{Tr}\;p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g),p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)]\\ &\qquad-\Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_2)g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g)]\\ &\qquad-\Im\mathrm{Tr}\;p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g),p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g)]\\ &\qquad-\Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_3)g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ \\ &\quad= -\Im\mathrm{Tr}\;p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ &\qquad-\Im\mathrm{Tr}\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_1)g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ &\qquad-\Im\mathrm{Tr}\;p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ &\qquad-\Im\mathrm{Tr}\,p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_2)g)]\\ &\qquad-\Im\mathrm{Tr}\;p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_3)\,g)[p_{\mathfrak{u}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ &\qquad-\Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{u}_2^+}(x_3)g)[p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g),p_{\mathfrak{b}_2^+}(g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g)]\\ \\ &\quad= -\Im\mathrm{Tr}\;g^{-1}p_{\mathfrak{u}_2^+}(x_3)g)[g^{-1}\,p_{\mathfrak{u}_2^+}(x_1)\,g,g^{-1}\,p_{\mathfrak{u}_2^+}(x_2)\,g]\\ &\quad= -\Im\mathrm{Tr}\,g^{-1}p_{\mathfrak{u}_2^+}(x_3)[p_{\mathfrak{u}_2^+}(x_1),p_{\mathfrak{u}_2^+}(x_2)]g\\ &\quad= -\Im\mathrm{Tr}\,p_{\mathfrak{u}_2^+}(x_3)[p_{\mathfrak{u}_2^+}(x_1),p_{\mathfrak{u}_2^+}(x_2)]\\ &\quad= 0.\end{aligned}$$

□

Remark 6.12. In the proof of previous Theorem, we have established that

$$T_e\pi_r(Y)([x_1]_{\mathfrak{b}^+_{1,2}}, [x_2]_{\mathfrak{b}^+_{1,2}}) \;= \Im\mathrm{Tr}\, Y[p_{\mathfrak{u}^+_2}(x_1), p_{\mathfrak{u}^+_2}(x_2)],$$

where x_1, $x_2 \in \mathrm{L}_{1,2}(\mathcal{H})$ and $Y \in \mathfrak{b}^+_{\mathrm{res}}(\mathcal{H})$. It follows that $T_e\pi_r$ is the dual map of

$$\begin{array}{rcl} \mathrm{L}_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H}) \times \mathrm{L}_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H}) & \to & \mathrm{L}_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H}) \\ ([x_1]_{\mathfrak{b}^+_{1,2}}, [x_2]_{\mathfrak{b}^+_{1,2}}) & \mapsto & [p_{\mathfrak{u}^+_2}(x_1), p_{\mathfrak{u}^+_2}(x_2)], \end{array} \tag{6.7}$$

which is well defined on $\mathrm{L}_{1,2}(\mathcal{H})/\mathfrak{b}^+_{1,2}(\mathcal{H})$ since $[p_{\mathfrak{u}^+_2}(x_1), p_{\mathfrak{u}^+_2}(x_2)] \in \mathrm{L}_1(\mathcal{H})$ for any x_1, $x_2 \in \mathrm{L}_{1,2}(\mathcal{H})$. Note that this bracket is continuous and extends the natural bracket of $\mathfrak{u}_{1,2}(\mathcal{H})$.

Along the same lines, one have the following Theorem :

Theorem 6.13. *Consider the Banach Lie group* $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$*, and*

(1) $\mathfrak{g}_+ := \mathrm{L}_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}(\mathcal{H}) \subset \mathfrak{u}^*_{\mathrm{res}}(\mathcal{H})$*,*
(2) $\mathbb{U} \subset T^*\,\mathrm{U}_{\mathrm{res}}(\mathcal{H})$*,* $\mathbb{U}_g = R^*_{g^{-1}}\mathfrak{g}_+$*,*
(3) $\tilde{\pi}_r \;: \mathrm{U}_{\mathrm{res}}(\mathcal{H}) \to \Lambda^2\mathfrak{g}^*_+(\mathfrak{g}_+)$ *defined by*

$$\tilde{\pi}_r(g)([x_1]_{\mathfrak{u}_{1,2}(\mathcal{H})}, [x_2]_{\mathfrak{u}_{1,2}(\mathcal{H})}) = \Im\mathrm{Tr}\,(g^{-1}\, p_{\mathfrak{b}^+_2}(x_1)\, g)\left[p_{\mathfrak{u}_2}(g^{-1}\, p_{\mathfrak{b}^+_2}(x_2)\, g)\right],$$

(4) $\tilde{\pi}(g) = R^{**}_g\tilde{\pi}_r(g)$*.*

Then $(\mathrm{U}_{\mathrm{res}}(\mathcal{H}), \mathbb{U}, \pi)$ *is a Banach Poisson-Lie group.*

7. Bruhat-Poisson structure of the restricted Grassmannian

7.1. **Banach Poisson subgroups.** The following definition is identical as in the finite-dimensional case.

Definition 7.1. A Lie subgroup H of a Banach Poisson-Lie group G is called a **Poisson-Lie subgroup** if it is a Banach Poisson submanifold of G, i.e. if it carries a Poisson structure such that the inclusion map $\iota\;: H \hookrightarrow G$ is a Poisson map.

Proposition 7.2. *The Banach Lie group* $\mathrm{H} := \mathrm{U}(\mathcal{H}_+)\times\mathrm{U}(\mathcal{H}_-)$ *is a Poisson-Lie subgroup of* $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$*.*

Proof. It is clear that H is a Banach submanifold of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. Denote by $\mathfrak{h}$ its Lie algebra. Recall that $\mathbb{U}$ is the subbundle of $T^*\,\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ given by $\mathbb{U}_g = R^*_{g^{-1}}\mathfrak{g}_+$ where $\mathfrak{g}_+ := L_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}(\mathcal{H})$. Denote by $\langle\cdot,\cdot\rangle_{\mathfrak{u}_{\mathrm{res}}}$ the duality pairing between $\mathfrak{g}_+$ and $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$, and by $\mathfrak{h}^0$ the closed subspace of $\mathfrak{g}_+$ consisting of those covectors in $\mathfrak{g}_+$ which vanish on the closed subspace $\mathfrak{h}$ of $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$. Then the formula

$$\langle[\alpha]_{\mathfrak{h}^0}, X\rangle_{\mathfrak{h}} := \langle\alpha, X\rangle_{\mathfrak{u}_{\mathrm{res}}},$$

where $[\alpha]_{\mathfrak{h}^0}$ denotes the class of $\alpha \in i^*\mathfrak{g}_+$ in $i^*\mathfrak{g}_+/\mathfrak{h}^0$ and where X belong to $\mathfrak{h}$, defines a duality pairing between $\mathbb{H}_e := i^*\mathfrak{g}_+/\mathfrak{h}^0$ and $\mathfrak{h}$. It follows that $\mathbb{H} := i^*\mathbb{U}/(TH)^0$ is a subbundle of T^*H in duality with TH. Recall that the Poisson tensor on $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ is defined as follows

$$\tilde{\pi}_r(h)(\alpha,\beta) = \Im\mathrm{Tr}\,(h^{-1}p_{\mathfrak{b}^+_2}(x_1)h)\left[p_{\mathfrak{u}^+_2}(h^{-1}p_{\mathfrak{b}^+_2}(x_2)h)\right]$$

where α, β in $\mathfrak{g}_+ = L_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}$ and x_1, $x_2 \in L_{1,2}(\mathcal{H})$ are such that $\alpha = [x_1]_{\mathfrak{u}_{1,2}}$ and $\beta = [x_2]_{\mathfrak{u}_{1,2}}$. Note that an element $x_2 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in L_{1,2}(\mathcal{H})$ belongs to $\mathfrak{h}^0$ if and only if $A \in \mathfrak{u}_1(\mathcal{H})$ and $D \in \mathfrak{u}_1(\mathcal{H})$. In that case, one has

$$x_2 = \left(\begin{smallmatrix} A & -C^* \\ C & D \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & B+C^* \\ 0 & 0 \end{smallmatrix}\right),$$

with $p_{\mathfrak{u}_2^+}(x_2) = \left(\begin{smallmatrix} A & -C^* \\ C & D \end{smallmatrix}\right)$ and $p_{\mathfrak{b}_2^+}(x_2) = \left(\begin{smallmatrix} 0 & B+C^* \\ 0 & 0 \end{smallmatrix}\right)$. Note also that for any $h = \left(\begin{smallmatrix} h_1 & 0 \\ 0 & h_2 \end{smallmatrix}\right) \in \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$, one has

$$h^{-1} p_{\mathfrak{b}_2^+}(x_2) h = \left(\begin{smallmatrix} 0 & h_1^{-1}(B+C^*)h_2 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{b}_2^+(\mathcal{H}).$$

It follows that $\tilde{\pi}_r(h)(\cdot, \beta) = 0$ whenever $\beta \in \mathfrak{h}^0$. By skew-symmetry of $\tilde{\pi}_r$, one also has $\tilde{\pi}_r(h)(\alpha, \cdot) = 0$ whenever $\alpha \in \mathfrak{h}^0$. This allows to define the following map

$$\Pi_r \ : H \to \Lambda^2 \mathbb{H}_e^*(\mathbb{H}_e)$$

by

$$\Pi_r(h)([\alpha]_{\mathfrak{h}^0}, [\beta]_{\mathfrak{h}^0}) := \tilde{\pi}_r(h)(\alpha, \beta)$$

for α, β in $\mathfrak{g}_+ = L_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}$. Set $\Pi := R_g^* \Pi_r$. The Jacobi identity for Π follows from the Jacobi identity for $\tilde{\pi}$. By construction, the injection $\iota : H \hookrightarrow \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ is a Poisson map. □

7.2. **The restricted Grassmannian as a quotient Poisson homogeneous space.**

Theorem 7.3. *The restricted Grassmannian* $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) = \mathrm{U}_{\mathrm{res}}(\mathcal{H})/\,\mathrm{U}(\mathcal{H}_+)\times\mathrm{U}(\mathcal{H}_-)$ *carries a natural Poisson structure* $(\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}), T^*\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}), \pi_{Gr_{\mathrm{res}}})$ *such that :*

(1) the canonical projection $p \ : \mathrm{U}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ *is a Poisson map,*
(2) the natural action $\mathrm{U}_{\mathrm{res}}(\mathcal{H}) \times \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ *by left translations is a Poisson map.*

Proof. (1) The tangent space at $eH \in \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) = \mathrm{U}_{\mathrm{res}}(\mathcal{H})/\,\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ can be identified with the quotient Banach space $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})/\left(\mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)\right)$ which is isomorphic to the Hilbert space

$$\mathfrak{m} := \{\left(\begin{smallmatrix} 0 & A \\ -A^* & 0 \end{smallmatrix}\right) \in \mathfrak{u}_2(\mathcal{H})\}.$$

The duality pairing between $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$ and $\mathfrak{g}_+ = L_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}(\mathcal{H})$ induces a strong duality pairing between the quotient space $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})/\left(\mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)\right) = \mathfrak{m}$ and $\mathfrak{h}^0 \subset \mathfrak{g}_+$. For $\alpha, \beta \in T^*_{gH}\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, identify $p^*\alpha \in T^*_g \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ with an element $L^*_{g^{-1}}x_1$ in $L^*_{g^{-1}}\mathfrak{h}^0$, and $p^*\beta$ with $L^*_{g^{-1}}x_2 \in L^*_{g^{-1}}\mathfrak{h}^0$. Define

$$\pi_{Gr_{\mathrm{res}}}(gH)(\alpha, \beta) \ = \tilde{\pi}_g(p^*\alpha, p^*\beta).$$

We have to check that the right hand side is invariant by the natural right action of H on $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, which induces an action of H on forms in $T^*_g\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ by $\gamma \to R^*_{h^{-1}}\gamma \in T^*_{gh}\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. In other words, we have to check that

$$\tilde{\pi}_g((p^*\alpha)_g, (p^*\beta)_g) = \tilde{\pi}_{gh}(R^*_{h^{-1}}(p^*\alpha)_g, R^*_{h^{-1}}(p^*\beta)_g) \tag{7.1}$$

$$\Leftrightarrow \tilde{\pi}_g(L^*_{g^{-1}}x_1, L^*_{g^{-1}}x_2) = \tilde{\pi}_{gh}(R^*_{h^{-1}}L^*_{g^{-1}}x_1, R^*_{h^{-1}}L^*_{g^{-1}}x_2)$$

$$\Leftrightarrow \ \tilde{\pi}_r(g)(\mathrm{Ad}^*_{g^{-1}}x_1, \mathrm{Ad}^*_{g^{-1}}x_2) = \tilde{\pi}_r(gh)(R^*_{gh}R^*_{h^{-1}}L^*_{g^{-1}}x_1, R^*_{gh}R^*_{h^{-1}}L^*_{g^{-1}}x_2)$$

Note that $R^*_{gh}\gamma(X) = \gamma(R_{gh}X) = \gamma(Xgh) = R^*_h\gamma(Xg) = R^*_g R^*_h\gamma(X)$. Therefore $R^*_{gh} = R^*_g R^*_h$. It follows that (7.1) is equivalent to

$$\tilde{\pi}_r(g)(\mathrm{Ad}^*_{g^{-1}}x_1, \mathrm{Ad}^*_{g^{-1}}x_2) = \tilde{\pi}_r(gh)(\mathrm{Ad}^*_{g^{-1}}x_1, \mathrm{Ad}^*_{g^{-1}}x_2)$$

By the cocycle identity $\tilde{\pi}_r(gh) = \mathrm{Ad}(g)^{**}\tilde{\pi}_r(h) + \tilde{\pi}_r(g)$, one has

$$\tilde{\pi}_r(gh)(\mathrm{Ad}^*_{g^{-1}}x_1, \mathrm{Ad}^*_{g^{-1}}x_2) = \tilde{\pi}_r(h)(\mathrm{Ad}^*_g\mathrm{Ad}^*_{g^{-1}}x_1, \mathrm{Ad}^*_g\mathrm{Ad}^*_{g^{-1}}x_2) + \tilde{\pi}_r(g)(\mathrm{Ad}^*_{g^{-1}}x_1, \mathrm{Ad}^*_{g^{-1}}x_2)$$

Since $\tilde{\pi}_r(h)$ vanishes on $\mathfrak{h}^0$, one has

$$\tilde{\pi}_r(h)(\mathrm{Ad}^*_{h^{-1}}x_1, \mathrm{Ad}^*_{h^{-1}}x_2) = 0,$$

therefore equation (7.1) is satisfied. The Jacobi identity of $\pi_{Gr_{\mathrm{res}}}$ follows from the Jacobi identity of $\tilde{\pi}$. Moreover p is a Poisson map by construction.

(2) Consider the action

$$\begin{array}{rcll} a_U & : & \mathrm{U}_{\mathrm{res}}(\mathcal{H}) \times \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) & \to & \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \\ & & (g_1, gH) & \mapsto & g_1 gH \end{array}$$

by left translations. Note that the tangent map to a_U is given by

$$\begin{array}{rcll} T_{(g_1,gH)}a_U & : & T_{g_1}\mathrm{U}_{\mathrm{res}}(\mathcal{H}) \oplus T_{gH}\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) & \to & T_{g_1gH}\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}) \\ & & (X_{g_1}, X_{gH}) & \mapsto & p_*[(R_g)_*X_{g_1}] + (L_{g_1})_*X_{gH}. \end{array}$$

Therefore, for any $\alpha \in T^*_{g_1gH}\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$,

$$\begin{array}{rl} \alpha \circ T_{(g_1,gH)}a_U(X_{g_1}, X_{gH}) & = \alpha(p_*[(R_g)_*X_{g_1}]) + \alpha((L_{g_1})_*X_{gH}) \\ & = R^*_g p^*\alpha(X_{g_1}) + L^*_{g_1}\alpha(X_{gH}). \end{array}$$

In other words

$$\alpha \circ T_{(g_1,gH)}a_U = R^*_g p^*\alpha + L^*_{g_1}\alpha,$$

where $R^*_g p^*\alpha \in T_{g_1}\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and $L^*_{g_1}\alpha \in T_{gH}\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$. In order to show that a_U is a Poisson map, we have to show that

(a) for any $\alpha \in T^*_{g_1gH}\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, the covector $R^*_g p^*\alpha$ belongs to

$$\mathbb{U}_{g_1} = R^*_{(g_1)^{-1}}L_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}(\mathcal{H}),$$

(b) the Poisson tensors $\tilde{\pi}$ and $\pi_{Gr_{\mathrm{res}}}$ are related by

$$(\pi_{Gr_{\mathrm{res}}})_{g_1gH}(\alpha, \beta) = \tilde{\pi}_{g_1}(R^*_g p^*\alpha, R^*_g p^*\beta) + (\pi_{Gr_{\mathrm{res}}})_{gH}(L^*_{g_1}\alpha, L^*_{g_1}\beta).$$

For point (a), let us show that for $\alpha \in T^*_{g_1gH}\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, and $g_1, g \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$, one has $R^*_{g_1}R^*_g p^*\alpha \in L_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}(\mathcal{H})$. Recall that $p^*\alpha$ can by identified with an element $L^*_{(g_1g)^{-1}}x_1$ where $x_1 \in \mathfrak{h}^0$. Therefore $R^*_{g_1}R^*_g p^*\alpha = \mathrm{Ad}^*_{(g_1g)^{-1}}x_1$. For $X \in T_e\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, one has

$$R^*_{g_1}R^*_g p^*\alpha(X) = \Im\mathrm{Tr}\, x_1\mathrm{Ad}_{(g_1g)^{-1}}(X) = \Im\mathrm{Tr}\, x_1(g_1g)^{-1}Xg_1g = \Im\mathrm{Tr}\, g_1gx_1(g_1g)^{-1}X.$$

Since $g_1gx_1(g_1g)^{-1} \in L_{1,2}(\mathcal{H})$ for any $g_1, g \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$ and $x_1 \in L_{1,2}(\mathcal{H})$, it follows that $R^*_{g_1}R^*_g p^*\alpha \in L_{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}(\mathcal{H})$.

In order to prove (b), note that for α, $\beta \in T^*_{g_1gH}\,\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, one has

$$\begin{aligned}(\pi_{Gr_{\mathrm{res}}})_{g_1gH}(\alpha,\beta) &= \tilde{\pi}_{g_1g}(p^*\alpha,p^*\beta) = \tilde{\pi}_r(g_1g)(R^*_{g_1g}p^*\alpha, R^*_{g_1g}p^*\beta)\\ &= \mathrm{Ad}(g_1)^*\tilde{\pi}_r(g)(R^*_{g_1g}p^*\alpha, R^*_{g_1g}p^*\beta) + \tilde{\pi}_r(g_1)(R^*_{g_1g}p^*\alpha, R^*_{g_1g}p^*\beta)\\ &= \tilde{\pi}_r(g)(L^*_{g_1}R^*_gp^*\alpha, L^*_{g_1}R^*_gp^*\beta) + \tilde{\pi}_r(g_1)(R^*_{g_1}R^*_gp^*\alpha, R^*_{g_1}R^*_gp^*\beta)\\ &= \tilde{\pi}_r(g)(R^*_gL^*_{g_1}p^*\alpha, R^*_gL^*_{g_1}p^*\beta) + \tilde{\pi}(g_1)(R^*_gp^*\alpha, R^*_gp^*\beta)\\ &= \tilde{\pi}_g(L^*_{g_1}p^*\alpha, L^*_{g_1}p^*\beta) + \tilde{\pi}(g_1)(R^*_gp^*\alpha, R^*_gp^*\beta)\\ &= \tilde{\pi}_g(p^*L^*_{g_1}\alpha, p^*L^*_{g_1}\beta) + \tilde{\pi}(g_1)(R^*_gp^*\alpha, R^*_gp^*\beta)\\ &= (\pi_{Gr_{\mathrm{res}}})_{gH}(L^*_{g_1}\alpha, L^*_{g_1}\beta) + \tilde{\pi}(g_1)(R^*_gp^*\alpha, R^*_gp^*\beta)\\ &= (\pi_{Gr_{\mathrm{res}}})_{gH}(L^*_{g_1}\alpha, L^*_{g_1}\beta) + \tilde{\pi}(g_1)(R^*_gp^*\alpha, R^*_gp^*\beta),\end{aligned}$$

where we have used the cocycle identity.

□

8. Dressing actions of $\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$ on $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, KdV hierarchy and Schubert cells

8.1. **Relation between the restricted Grassmannian and the KdV hierarchy.** Let us recall the construction of G. Segal and G. Wilson detailled in [SW85] which gives a correspondance between some elements of the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ and solutions of the KdV hierarchy.

We will need to introduce some additional notation. In this section, $\mathcal{H} = L^2(\mathbb{S}^1,\mathbb{C})$, and the inner product of two elements f and g in $L^2(\mathbb{S}^1,\mathbb{C})$ reads $\langle f,g\rangle = \int_{\mathbb{S}^1}\overline{f(z)}g(z)d\mu(z)$, where $d\mu(z)$ denotes the Lebesgue mesure on the circle. Set $\mathcal{H}_+ = \mathrm{span}\{z^n, n\geq 0\}$ and $\mathcal{H}_- = \mathrm{span}\{z^n, n<0\}$. Let Γ_+ be the group of real-analytic functions $g\ :\mathbb{S}^1\to\mathbb{C}^*$, which extend to holomorphic functions g from the unit disc $\mathbb{D} = \{z\in\mathbb{C}\ :|z|\leq 1\}$ to $\mathbb{C}^*$, satisfying $g(0)=1$. Any such function $g\in\Gamma^+$ can be written $g = e^f$, where f is a holomorphic function on $\mathbb{D}$ such that $f(0)=0$.

Proposition 8.1. *The group Γ_+ acts by multiplication operators on $\mathcal{H}$ and $\Gamma_+\subset\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$.*

Proof. By Proposition 2.3 in [SW85], $\Gamma^+\subset\mathrm{GL}_{\mathrm{res}}(\mathcal{H}) := \mathrm{GL}(\mathcal{H})\cap\mathrm{L}_{\mathrm{res}}(\mathcal{H})$. Since $g\in\Gamma^+$ is holomorphic in $\mathbb{D}$ and satisfies $g(0)=1$, the Fourier decomposition of g reads $g(z) = 1+\sum_{k>0}g_kz^k$. Therefore $g(z)\cdot z^n = z^n+\sum_{k>0}g_kz^{k+n}$. It follows that the multiplication operator by g is a upper triangular operator $M_g\in\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$, with diagonal elements equal to 1. □

Following [SW85], we will denote by $\mathrm{Gr}^{(n)}$ the subset of the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ given by

$$\mathrm{Gr}^{(n)} = \{w\in\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})\ :z^nW\subset W\}.$$

Moreover, given a subspace $W\in\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, we set

$$\Gamma^+_W = \{g\in\Gamma_+\ :g^{-1}W\cap\mathcal{H}_- = \{0\}\}.$$

Let us know recall the following Proposition :

Proposition 8.2 (Proposition 5.1 in [SW85])**.** *For each $W\in\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$, there is a unique function $\Phi_W(g,z)$ called the Baker function of W, defined for $g\in\Gamma^+_W$ and $z\in\mathbb{S}^1$, such that*

(i) $\Phi_W(g,\cdot)\in W$ for each fixed $g\in\Gamma^+_W$

(ii) Φ_W *has the form*

$$\Phi_W = g(z)(1 + \sum_1^\infty a_i(g) z^{-i}).$$

The coefficients a_i *are analytic functions on* Γ^+_W *and extend to meromorphic functions on the whole of* Γ^+.

Since any $g \in \Gamma_+$ can be written uniquely as $g(z) = \exp(xz + t_2 z^2 + t_3 z^3 + \dots)$, the Baker function of $W \in \mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ as the following expression :

$$\Phi_W = \exp(xz + t_2 z^2 + t_3 z^3 + \dots)(1 + \sum_1^\infty a_i(g) z^{-i}).$$

Now the following Proposition assigns to $W \in \mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ a hierarchy of differential operators P_r :

Proposition 8.3 (Proposition 5.5 in [SW85]). *Set* $D = \frac{\partial}{\partial x}$. *For each integer* $r \geq 2$, *there is a unique differential operator* P_r *of the form*

$$P_r = D^r + p_{r2} D^{r-2} + \dots + p_{r,r-1} D + p_{rr}$$

such that

$$\frac{\partial \Phi_W}{\partial t_r} = P_r \Phi_W.$$

Denote by $\mathcal{C}^{(n)}$ the space of all operators P_n associated to subspaces W in $\mathrm{Gr}^{(n)}$ and evaluated at $t_2 = t_3 = \dots = 0$. Then

Proposition 8.4 (Proposition 5.13 in [SW85]). *The action of* Γ_+ *on* $\mathrm{Gr}^{(n)}$ *induces an action on the space* $\mathcal{C}^{(n)}$. *For* $r \geq 1$, *the flow* $W \mapsto \exp(t_r z^r) W$ *on* $\mathrm{Gr}^{(n)}$ *induces the* r*-th KdV flow on* $\mathcal{C}^{(n)}$.

8.2. **Dressing action of** $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$ **on** $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H})$**.** Recall that for $r \geq 1$, the multiplication operator by $\exp(t_r z^r)$ belongs to $\Gamma_+ \subset \mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$. The next Theorem shows that the action of $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$ on $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ is a Poisson map, where $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$ is endowed with the Banach Poisson-Lie group structure defined in section 6, and where $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ is endow with the Bruhat-Poisson structure defined in section 7.

Theorem 8.5. *The following right action of* $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$ *on* $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) = \mathrm{GL}_{\mathrm{res}}(\mathcal{H}) / \mathrm{P}_{\mathrm{res}}(\mathcal{H})$ *by dressing transformations is a Poisson map :*

$$\begin{array}{rcl} a_B \; : \; \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \times \mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H}) & \to & \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \\ (g\,\mathrm{P}_{\mathrm{res}}, b) & \mapsto & (b^{-1}g)\,\mathrm{P}_{\mathrm{res}}(\mathcal{H}). \end{array}$$

Proof. The tangent map to the action a_B reads

$$\begin{array}{rcl} T_{(gH,b)} a_B \; : \; T_{gH}\,\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \oplus T_b\,\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H}) & \to & T_{b^{-1}gP_{\mathrm{res}}}\,\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \\ (X_{gH}, X_b) & \mapsto & (L_{(b^{-1})})_* X_{gH} - p_*(R_g)_*(b^{-1} X_b b^{-1}). \end{array}$$

Therefore, for any $\alpha \in T^*_{b^{-1}gP_{\mathrm{res}}}\,\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$,

$$\begin{array}{rl} \alpha \circ T_{(gH,b)} a_B(X_{gH}, X_b) & = \alpha((L_{(b^{-1})})_* X_{gH}) - \alpha(p_*(R_g)_* b^{-1} X_b b^{-1}) \\ & = L^*_{b^{-1}}\alpha(X_{gH}) - R^*_{b^{-1}} L^*_{b^{-1}} R^*_g p^* \alpha(X_b), \end{array}$$

and

$$\alpha \circ T_{(gH,b)} a_B = L^*_{b^{-1}}\alpha - R^*_{b^{-1}} L^*_{b^{-1}} R^*_g p^* \alpha,$$

where $L^*_{b^{-1}}\alpha \in T^*_{gH}\,\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ and $R^*_{b^{-1}} L^*_{b^{-1}} R^*_g p^* \alpha \in T^*_b\,\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$.

(a) Let us show that for any $\alpha \in T^*_{b^{-1}gP_{\rm res}} \operatorname{Gr}_{\rm res}(\mathcal{H})$ and any $b \in \mathrm{B}^{\pm}_{\rm res}(\mathcal{H})$, the form $R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\alpha$ belongs to $\mathbb{B}_b = R^*_{b^{-1}}L_{1,2}(\mathcal{H})/\mathfrak{b}_{1,2}(\mathcal{H})$. Recall that α can be identified with an element $L^*_{(b^{-1}g)^{-1}}x_1$ where $x_1 \in \mathfrak{h}^0$. For $X \in T_e \mathrm{B}^{\pm}_{\rm res}(\mathcal{H})$, one has

$$\begin{aligned} L^*_{b^{-1}}R^*_g p^*\alpha(X) &= \alpha(p_* R_{g^*}(L_{b^{-1}})_* X) = \Im \mathrm{Tr}\, x_1 (L_{g^{-1}b})_* p_* R_{g^*}(L_{b^{-1}})_* X) \\ &= \Im\mathrm{Tr}\, x_1 p_*(\mathrm{Ad}(g^{-1})X) = \Im\mathrm{Tr}\, p_{\mathfrak{b}_2^+}(x_1) g^{-1} X g \\ &= \Im\mathrm{Tr}\, g p_{\mathfrak{b}_2^+}(x_1) g^{-1} X. \end{aligned}$$

Recall that for $x_1 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{h}^0$, $p_{\mathfrak{b}_2^+}(x_1) = \left(\begin{smallmatrix} 0 & B+C^* \\ 0 & 0 \end{smallmatrix}\right)$. Since for any $g \in GL_{\rm res}(\mathcal{H})$ and any $x_1 \in \mathfrak{h}^0$, $g p_{\mathfrak{b}_2^+}(x_1) g^{-1} \in L_{1,2}(\mathcal{H})$, the form $R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\alpha$ belongs to $\mathbb{B}_b$.

(b) Let us show that the Poisson tensors π and $\pi_{Gr_{\rm res}}$ are related by

$$(\pi_{Gr_{\rm res}})_{b^{-1}gP_{\rm res}}(\alpha,\beta) = (\pi_{Gr_{\rm res}})_{gH}(L^*_{b^{-1}}\alpha, L^*_{b^{-1}}\beta) + \pi_b(R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\alpha, R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\beta).$$

One has

$$\begin{aligned} \pi_b(R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\alpha, R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\beta) &= \pi_r(b)([g p_{\mathfrak{b}_2^+}(x_1)g^{-1}]_{\mathfrak{b}^+_{1,2}}, [g p_{\mathfrak{b}_2^+}(x_2)g^{-1}]_{\mathfrak{b}^+_{1,2}}) \\ &= \Im\mathrm{Tr}\left(b^{-1}p_{\mathfrak{u}_2^+}(g p_{\mathfrak{b}_2^+}(x_1)g^{-1})b\right)\left[p_{\mathfrak{b}_2^+}(b^{-1}p_{\mathfrak{u}_2^+}(g p_{\mathfrak{b}_2^+}(x_2)g^{-1})b)\right] \\ &= \Im\mathrm{Tr}\, p_{\mathfrak{u}_2^+}(g p_{\mathfrak{b}_2^+}(x_1)g^{-1})b\left[p_{\mathfrak{b}_2^+}(b^{-1}p_{\mathfrak{u}_2^+}(g p_{\mathfrak{b}_2^+}(x_2)g^{-1})b)\right]b^{-1} \\ &= \Im\mathrm{Tr}\,(b^{-1}g p_{\mathfrak{b}_2^+}(x_1)g^{-1}b)\left[p_{\mathfrak{b}_2^+}(b^{-1}p_{\mathfrak{u}_2^+}(g p_{\mathfrak{b}_2^+}(x_2)g^{-1})b)\right] \\ &= \Im\mathrm{Tr}\,(b^{-1}g p_{\mathfrak{b}_2^+}(x_1)g^{-1}b)\left[p_{\mathfrak{b}_2^+}(b^{-1}g p_{\mathfrak{b}_2^+}(x_2)g^{-1}b)\right] \\ &\quad -\Im\mathrm{Tr}\,(b^{-1}g p_{\mathfrak{b}_2^+}(x_1)g^{-1}b)\left[p_{\mathfrak{b}_2^+}(b^{-1}p_{\mathfrak{b}_2^+}(g p_{\mathfrak{b}_2^+}(x_2)g^{-1})b)\right] \end{aligned}$$

Therefore

$$\begin{aligned} &\pi_b(R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\alpha, R^*_{b^{-1}}L^*_{b^{-1}}R^*_g p^*\beta) = \\ &\Im\mathrm{Tr}\,(b^{-1}g p_{\mathfrak{b}_2^+}(x_1)g^{-1}b)\left[p_{\mathfrak{b}_2^+}(b^{-1}g p_{\mathfrak{b}_2^+}(x_2)g^{-1}b)\right] - \Im\mathrm{Tr}\,(g p_{\mathfrak{b}_2^+}(x_1)g^{-1})\left[p_{\mathfrak{b}_2^+}(g p_{\mathfrak{b}_2^+}(x_2)g^{-1})\right]. \end{aligned} \tag{8.1}$$

On the other hand

$$\begin{aligned} (\pi_{Gr_{\rm res}})_{gH}(L^*_{b^{-1}}\alpha, L^*_{b^{-1}}\beta) &= \tilde{\pi}_r(g)([g p_{\mathfrak{b}_2^+}(x_1)g^{-1}], [g p_{\mathfrak{b}_2^+}(x_2)g^{-1}]) \\ &= \Im\mathrm{Tr}\,(g^{-1}p_{\mathfrak{b}_2^+}(g p_{\mathfrak{b}_2^+}(x_1)g^{-1})g)\left[p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{b}_2^+}(g p_{\mathfrak{b}_2^+}(x_1)g^{-1})g)\right] \\ &= \Im\mathrm{Tr}\, p_{\mathfrak{b}_2^+}(x_1)\left[p_{\mathfrak{u}_2^+}(g^{-1}p_{\mathfrak{b}_2^+}(g p_{\mathfrak{b}_2^+}(x_1)g^{-1})g)\right] \\ &= \Im\mathrm{Tr}\, p_{\mathfrak{b}_2^+}(x_1)(g^{-1}p_{\mathfrak{b}_2^+}(g p_{\mathfrak{b}_2^+}(x_1)g^{-1})g \\ &= \Im\mathrm{Tr}\, g p_{\mathfrak{b}_2^+}(x_1)g^{-1}p_{\mathfrak{b}_2^+}(g p_{\mathfrak{b}_2^+}(x_1)g^{-1}) \end{aligned}$$

which is the second term in the right hand side of equation (8.1) with the opposite sign. Moreover, since

$$\operatorname{Gr}_{\rm res}(\mathcal{H}) = \operatorname{GL}_{\rm res}(\mathcal{H})/\operatorname{P}_{\rm res}(\mathcal{H}) = \operatorname{U}_{\rm res}(\mathcal{H})/\left(\operatorname{U}(\mathcal{H}_+)\times\operatorname{U}(\mathcal{H}_-)\right)$$

there exist $g_1 \in \mathrm{U}_{\rm res}(\mathcal{H})$ and $p_1 \in P_{\rm res}(\mathcal{H})$ such that $b^{-1}g = g_1 p_1$. In fact, the couple (g_1, p_1) is defined modulo the right action by H given by $(g_1,p_1)\cdot h = (g_1 h, h^{-1}p_1)$. It follows that the first term in the right hand side of equation (8.1) reads

$$\begin{aligned} &\Im\mathrm{Tr}\,(b^{-1}g p_{\mathfrak{b}_2^+}(x_1)g^{-1}b)\left[p_{\mathfrak{b}_2^+}(b^{-1}g p_{\mathfrak{b}_2^+}(x_2)g^{-1}b)\right] \\ &\quad = \Im\mathrm{Tr}\,(g_1 p_1 p_{\mathfrak{b}_2^+}(x_1)p_1^{-1}g_1^{-1})\left[p_{\mathfrak{b}_2^+}(g_1 p_1 p_{\mathfrak{b}_2^+}(x_2)p_1^{-1}g_1^{-1})\right] \end{aligned}$$

Recall that for any $x_1 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{h}^0$, one has

$$x_1 = \left(\begin{smallmatrix} A & -C^* \\ C & D \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & B+C^* \\ 0 & 0 \end{smallmatrix}\right),$$

with $p_{\mathfrak{u}_2^+}(x_1) = \left(\begin{smallmatrix} A & -C^* \\ C & D \end{smallmatrix}\right)$ and $p_{\mathfrak{b}_2^+}(x_1) = \left(\begin{smallmatrix} 0 & B+C^* \\ 0 & 0 \end{smallmatrix}\right)$. Note that for any $p_1 = \left(\begin{smallmatrix} P_1 & P_2 \\ 0 & P_3 \end{smallmatrix}\right) \in P_{\text{res}}(\mathcal{H})$, one has

$$p_1^{-1} = \begin{pmatrix} P_1^{-1} & -P_1^{-1}P_2P_3^{-1} \\ 0 & P_3^{-1} \end{pmatrix} \in P_{\text{res}}(\mathcal{H}),$$

and

$$p_1 p_{\mathfrak{b}_2^+}(x_1) p_1^{-1} = \begin{pmatrix} 0 & P_1(B+C^*)P_3^{-1} \\ 0 & 0 \end{pmatrix} \in \mathfrak{b}_2^+(\mathcal{H}).$$

Therefore

$$\begin{aligned}
&\Im \mathrm{Tr}\,(b^{-1}gp_{\mathfrak{b}_2^+}(x_1)g^{-1}b)\left[p_{\mathfrak{b}_2^+}(b^{-1}gp_{\mathfrak{b}_2^+}(x_2)g^{-1}b)\right]\\
&= \Im \mathrm{Tr}\,(g_1p_1p_{\mathfrak{b}_2^+}(x_1)p_1^{-1}g_1^{-1})\left[p_{\mathfrak{b}_2^+}(g_1p_1p_{\mathfrak{b}_2^+}(x_2)p_1^{-1}g_1^{-1})\right]\\
&= \tilde{\pi}_r(g_1)([g_1p_1p_{\mathfrak{b}_2^+}(x_1)p_1^{-1}g_1^{-1}], [g_1p_1p_{\mathfrak{b}_2^+}(x_2)p_1^{-1}g_1^{-1}])\\
&= \tilde{\pi}_r(g_1)([b^{-1}gp_{\mathfrak{b}_2^+}(x_1)g^{-1}b], [b^{-1}gp_{\mathfrak{b}_2^+}(x_2)g^{-1}b])\\
&= (\pi_{Gr_{\text{res}}})_{g_1H}(\alpha,\beta) = (\pi_{Gr_{\text{res}}})_{b^{-1}gP_{\text{res}}}(\alpha,\beta).
\end{aligned}$$

□

8.3. **Schubert cells of the restricted Grassmannian.** Let us recall some geometric facts about the restricted Grassmannian that were established in [PS88], Chapter 7. The restricted Grassmannian admits a stratification $\{\Sigma_S, S \in \mathcal{S}\}$ as well as a decomposition into Schubert cells $\{\mathcal{C}_S, S \in \mathcal{S}\}$, which are dual to each other in the following sense :

(i) the same set $\mathcal{S}$ indexes the cells $\{\mathcal{C}_S\}$ and the strata $\{\Sigma_S\}$;
(ii) the dimension of $\mathcal{C}_S$ is the codimension of Σ_S;
(iii) $\mathcal{C}_S$ meets Σ_S transversally in a single point, and meets no other stratum of the same codimension.

A element S of the set $\mathcal{S}$ is a subset of $\mathbb{Z}$, which is bounded from below and contains all sufficiently large integers. Given $S \in \mathcal{S}$, define the subspace $\mathcal{H}_S$ of the restricted Grassmannian $\mathrm{Gr}_{\text{res}}(\mathcal{H})$ by :

$$\mathcal{H}_S = \overline{\text{span}\{z^s, s \in S\}}.$$

Recall the following Proposition :

Proposition 8.6 (Proposition 7.1.6 in [PS88])**.** *For any $W \in \mathrm{Gr}_{\text{res}}(\mathcal{H})$ there is a set $S \in \mathcal{S}$ such that the orthogonal projection $W \to \mathcal{H}_S$ is an isomorphism. In other words the sets $\{\mathcal{U}_S, S \in \mathcal{S}\}$, where*

$$\mathcal{U}_S = \{W \in \mathrm{Gr}_{\text{res}}(\mathcal{H}), \textit{the orthogonal projection } W \to \mathcal{H}_S \textit{ is an isomorphism}\},$$

form an open covering of $\mathrm{Gr}_{\text{res}}(\mathcal{H})$.

Following [PS88], let us introduce the Banach Lie groups $\mathrm{N}^+_{\text{res}}(\mathcal{H})$ and $\mathrm{N}^-_{\text{res}}(\mathcal{H})$:

$$\mathrm{N}^+_{\text{res}}(\mathcal{H}) = \{A \in \mathrm{GL}_{\text{res}}(\mathcal{H}), A(z^k\mathcal{H}_+) = z^k\mathcal{H}_+ \text{ and } (A - \mathrm{Id})(z^k\mathcal{H}_+) \subset z^{k-1}\mathcal{H}_+,\ \forall k \in \mathbb{Z}\},$$

$$\mathrm{N}^-_{\text{res}}(\mathcal{H}) = \{A \in \mathrm{GL}_{\text{res}}(\mathcal{H}), A(z^k\mathcal{H}_-) = z^k\mathcal{H}_- \text{ and } (A - \mathrm{Id})(z^k\mathcal{H}_-) \subset z^{k-1}\mathcal{H}_-,\ \forall k \in \mathbb{Z}\}.$$

In other words, the group $\mathrm{N}^\pm_{\text{res}}(\mathcal{H})$ is the subgroup of $\mathrm{B}^\pm_{\text{res}}(\mathcal{H})$ consisting of the triangular operators with respect to the basis $\{|n\rangle := z^n, n \in \mathbb{Z}\}$ which have only 1's on the diagonal.

Proposition 8.7. *The Banach Lie group* $\mathrm{N}^{\pm}_{\mathrm{res}}(\mathcal{H})$ *is a normal subgroup of* $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$ *and the quotient group* $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})/\mathrm{N}^{\pm}_{\mathrm{res}}(\mathcal{H})$ *is isomorphic to the group of bounded linear positive definite operators which are diagonal with respect to the orthonormal basis* $\{|z^k\rangle, k \in \mathbb{Z}\}$*.*

Proof. Follows from the decomposition of a operator in $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$ into the product of a diagonal operator and a operator in $\mathrm{N}^{\pm}_{\mathrm{res}}(\mathcal{H})$. □

Proposition 8.8. *(i) The cell* $\mathcal{C}_S$ *is the orbit of* $\mathcal{H}_S$ *under* $\mathrm{B}^{+}_{\mathrm{res}}(\mathcal{H})$*.*
(ii) The strata Σ_S *is the orbit of* $\mathcal{H}_S$ *under* $\mathrm{B}^{-}_{\mathrm{res}}(\mathcal{H})$*.*

Proof. It follows from Proposition 7.4.1 in [PS88], that the cell $\mathcal{C}_S$ is the orbit of $\mathcal{H}_S$ under $\mathrm{N}^{+}_{\mathrm{res}}(\mathcal{H})$. Symmetrically, it follows from Proposition 7.3.3 in [PS88], that the strata Σ_S is the orbit of $\mathcal{H}_S$ under $\mathrm{N}^{-}_{\mathrm{res}}(\mathcal{H})$. Since the diagonal part of an operator in $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$ acts trivially, one gets the same result replacing $\mathrm{N}^{\pm}_{\mathrm{res}}(\mathcal{H})$ by $\mathrm{B}^{\pm}_{\mathrm{res}}(\mathcal{H})$. □

Theorem 8.9. *The Schubert cells* $\{\mathcal{C}_S, S \in \mathcal{S}\}$ *are the symplectic leaves of* $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$*.*

Proof. The integrability of the characteristic distribution follows from Theorem 6 in [P12], since $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ is a Hilbert manifold. The fact that the symplectic leaves are the orbits of $\mathrm{B}^{+}_{\mathrm{res}}(\mathcal{H})$ follows from the construction as in the finite-dimensional case. We conclude using Proposition 8.8. □

Acknowledgment 1. The acknowledgments will be added after the reviewing process is completed.

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Notation	Explanation
$\mathcal{H}$	complex separable infinite-dimensional Hilbert space
$\mathbb{S}^1$	unit circle
$L^2(\mathbb{S}^1, \mathbb{C})$	complex-valued square-integrable functions
$\mathrm{L}_\infty(\mathcal{H})$	Banach space of bounded operators over $\mathcal{H}$
$\mathrm{L}_2(\mathcal{H})$	Hilbert space of Hilbert-Schmidt operators over $\mathcal{H}$
$\mathrm{L}_1(\mathcal{H})$	Banach space of trace-class operators over $\mathcal{H}$
$\mathrm{L}_{\mathrm{res}}(\mathcal{H})$	restricted Banach algebra defined by (2.1)
$\mathrm{L}_{1,2}(\mathcal{H})$	predual of $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$ defined by (2.2)
$\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$	restricted general linear group defined by (2.3)
$\mathrm{GL}_{1,2}(\mathcal{H})$	Banach Lie group with Lie algebra $\mathrm{L}_{1,2}(\mathcal{H})$
$\mathrm{GL}_2(\mathcal{H})$	Hilbert Lie group with Lie algebra $\mathrm{L}_2(\mathcal{H})$
$\mathfrak{u}(\mathcal{H})$	real Banach Lie algebra of skew-hermitian bounded operators over $\mathcal{H}$
$\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$	real Banach Lie algebra of skew-hermitian operators in $\mathrm{L}_{\mathrm{res}}(\mathcal{H})$, see (2.7)
$\mathfrak{u}_{1,2}(\mathcal{H})$	real Banach Lie algebra of skew-hermitian operators in $\mathrm{L}_{1,2}(\mathcal{H})$, see (2.8)
$\mathfrak{u}_2(\mathcal{H})$	real Hilbert Lie algebra of skew-hermitian Hilbert-Schmidt operators, see (2.9)
$\mathrm{U}(\mathcal{H})$	real Banach Lie group of unitary operators over $\mathcal{H}$
$\mathrm{U}_{\mathrm{res}}(\mathcal{H})$	restricted unitary group defined by (2.10)
$\mathrm{U}_{1,2}(\mathcal{H})$	real Banach Lie group with Lie algebra $\mathfrak{u}_{1,2}(\mathcal{H})$
$\mathrm{U}_2(\mathcal{H})$	real Hilbert Lie group with Lie algebra $\mathfrak{u}_2(\mathcal{H})$
$\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$	the restricted Grassmannian defined in section 2.8
$\mathrm{P}_{\mathrm{res}}(\mathcal{H})$	Parabolic subgroup of $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ defined by (2.13)
$\mathrm{L}_2(\mathcal{H})_-$	Lower triangular Hilbert-Schmidt operators over $\mathcal{H}$
$\mathrm{L}_2(\mathcal{H})_{++}$	Stricktly upper triangular Hilbert-Schmidt operators over $\mathcal{H}$
$\mathrm{L}_2(\mathcal{H})_+$	Upper triangular Hilbert-Schmidt operators over $\mathcal{H}$
$\mathrm{L}_2(\mathcal{H})_{--}$	Stricktly lower triangular Hilbert-Schmidt operators over $\mathcal{H}$
T_-	Lower triangular truncation defined by (2.14)
T_{++}	Stricktly upper triangular truncation defined by (2.15)
D	Diagonal truncation defined in (2.16)
$\mathfrak{b}_2^\pm(\mathcal{H}), \mathfrak{b}_{1,2}^\pm(\mathcal{H})$ and $\mathfrak{b}_{\mathrm{res}}^\pm$	triangular Banach Lie subalgebras defined in section 2.10
$\mathrm{B}_2^\pm(\mathcal{H})$, $\mathrm{B}_{1,2}^\pm(\mathcal{H})$, and $\mathrm{B}_{\mathrm{res}}^\pm(\mathcal{H})$	Triangular Banach Lie groups defined in section 2.11
$\mathcal{K}(\mathcal{H})$	Ideal of compact operators over $\mathcal{H}$
$L^{r,s}(\mathfrak{g}_-, \mathfrak{g}_+; \mathbb{K})$	the space of continuous multilinear maps from $\mathfrak{g}_- \times \cdots \times \mathfrak{g}_- \times \mathfrak{g}_+ \times \cdots \times \mathfrak{g}_+$ to $\mathbb{K}$, where $\mathfrak{g}_-$ is repeated r-times and $\mathfrak{g}_+$ is repeated s-times