QUEER POISSON BRACKETS

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ABSTRACT. We give a method to construct Poisson brackets $\{\,\cdot\,,\,\cdot\,\}$ on Banach manifolds M, for which the value of $\{f, g\}$ at some point $m \in M$ may depend on higher order derivatives of the smooth functions $f, g\colon M \to \mathbb{R}$, and not only on the first-order derivatives, as it is the case on all finite-dimensional manifolds. We discuss specific examples in this connection, as well as the impact on the earlier research on Poisson geometry of Banach manifolds. Those brackets are counterexamples to the claim that the Leibniz property for any Poisson bracket on a Banach manifold would imply the existence of a Poisson tensor for that bracket.

1. INTRODUCTION

The Poisson brackets in infinite-dimensional setting have played for a long time a significant role in various areas of mathematics including (classical and quantum) mechanics and integrable systems theory (see e.g. [Fad80, B00, AMR02, CM74]). However the rigorous approach to the notion of Poisson manifold in the context of Banach space is relatively recent (see [OR03]). It is

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known that the Poisson brackets on infinite-dimensional manifolds lack some of the properties known from the finite-dimensional case. It was shown for instance in [OR03] that the existence of Hamiltonian vector fields requires an additional condition on the Poisson tensor in the case of manifolds modelled on a non-reflexive Banach space (i.e. a Banach space E that is not canonically isomorphic to its second dual $E \subsetneq E^{**}$, where E^* denotes the topological dual of a Banach space). Another example of a new behaviour can be found in [Dit05] — a Poisson bracket defined only on a certain space of smooth functions might lead to an unbounded Poisson tensor. Moreover on some manifolds, Poisson brackets need not be local although as far as we know a counterexample is not known yet, see a related discussion in [CP12].

The aim of this paper is to prove by example still another phenomenon that is specific to Poisson geometry on an infinite dimensional manifold M, namely the existence of Poisson brackets of higher order. That is, Leibniz property does not ensure that the bracket depends only on the first-order derivatives of functions. The constructed Poisson brackets serve as a counterexample to the statements given in the literature (see [OR03] or subsequently [Ida11]), where it was claimed that the existence of a Poisson tensor Π follows from Leibniz property and skew symmetry of the Poisson bracket $\{\cdot, \cdot\}$, in particular for every $m \in M$ one could find a bounded bilinear functional $\Pi_m \colon T_m^* M \times T_m^* M \to \mathbb{R}$ satisfying

$$\{f, g\}(m) = \Pi_m(f'_m, g'_m)$$

where $f'_m, g'_m \in T_m^* M$ are the differentials of $f, g \in C^\infty(M)$ at point $m \in M$. There is a related fact in [AMR02, Thm. 4.2.16], but we show that it is not applicable here (see Proposition 2.6).

We prove the existence of Poisson brackets not given by Poisson tensors on the family of Banach sequence spaces l^p for $1 \le p \le 2$ and present an explicit example for $p = 2$. Such Poisson brackets are pathological and do not lead to the dynamics in the usual way, thus from the point of view of applications in physics one should explicitly assume the existence of Poisson tensor in the definition of Poisson Banach manifold.

In section 2 we investigate "queer operational tangent vectors", that is, derivations which are differential operators of order higher than 1 on spaces of smooth functions on the manifold. This notion was introduced with several results on their existence (including the examples on the Hilbert space) in [KM97]. We explore the case of queer vectors of order 2 on the family of Banach sequence spaces l^p for $1 \le p < \infty$.

Section 3 contains our main result, which shows a way to construct higher order Poisson brackets out of queer vector fields, and we illustrate the general result by a specific example on the Hilbert space. We conclude the paper with

a version of the definition of Banach Poisson manifold which clarifies the one introduced in [OR03]

All the Banach and Hilbert spaces considered in this paper are real. By manifold we will always mean a smooth real manifold modelled on a Banach space.

2. QUEER OPERATIONAL VECTOR FIELDS

There are two major approaches to tangent vectors, namely the kinematic one and the operational one. These approaches lead to the same notion for finite-dimensional manifolds, but this is no longer the case in infinite dimensions. A kinematic tangent vector to a Banach manifold M at a point $m \in M$ is an equivalence class of curves passing through that point (for precise definition see e.g. [AMR02]). On the other hand, an operational tangent vector is defined as a derivation acting in the space of germs of functions (see [KM97], [CP12]).

For any $m \in M$ we consider the set of all functions $f \colon U \to \mathbb{R}$ defined on an open neighborhood U of m. We define an equivalence relation in that set in the following way: two functions $f_1 \colon U_1 \to \mathbb{R}$ and $f_2 \colon U_2 \to \mathbb{R}$ are equivalent if there exists an open neighborhood $U \subset U_1 \cap U_2$ of m for which the restrictions of f_1 and f_2 to U coincide. Any equivalence class $[f]$ defined in this way is called a germ at the point $m \in M$. We denote the set of germs of all smooth functions at m by $C_m^\infty(M)$. We will drop the brackets in the notation of germ when there is no risk of confusion.

Definition 2.1. An **operational tangent vector** at point $m \in M$ is a linear map $\delta \colon C_m^\infty(M) \to \mathbb{R}$ satisfying Leibniz rule :

$$\delta(fg) = \delta f \, g(m) + f(m) \, \delta g. \qquad (2.1)$$

An **operational vector field** on M is a collection of maps $\delta_U \colon C^\infty(U) \to C^\infty(U)$ for each open set $U \subset M$, compatible with restrictions to open subsets and defining an operational tangent vector at every $m \in M$.

Let us denote by $L_k(T_mM; \mathbb{R})$ the space of bounded k-linear maps on T_mM with values in $\mathbb{R}$ and let $f_m^{(k)} \in L_k(T_mM; \mathbb{R})$ be the k-th differential of a function f at $m \in M$.

Definition 2.2. The operational tangent vector δ is of **order** n if it can be expressed in the form

$$\delta f = \sum_{k=1}^{n} \ell_k(f_m^{(k)}), \qquad (2.2)$$

with $\ell_k \colon L_k(T_mM; \mathbb{R}) \to \mathbb{R}$, and ℓ_n does not vanish identically on the subspace of symmetric n-linear maps in $L_k(T_mM; \mathbb{R})$. Otherwise the order of δ

is infinite. If only one term is present in the sum (2.2), δ is called homogeneous. The operational tangent vectors of order at least 2 are called **queer**.

The operational vector field δ is of **order at most** n if there exists a family of smooth sections ℓ_k of the bundle $\bigsqcup_{m \in M} (L_k(T_m M; \mathbb{R}))^*$ satisfying (2.2) at each $m \in M$.

The Leibniz rule (2.1) satisfied by δ implies certain algebraic conditions on functionals ℓ_k, see [KM97, 28.2].

By definition, operational tangent vectors of order n depend only on the nth jet of functions. The existence of infinite order operational tangent vectors is an open problem as far as we know.

Remark 2.3. Any kinematic tangent vector defines an operational tangent vector of order 1. On the other hand in the case of manifolds modelled on non-reflexive Banach spaces, operational tangent vectors of order 1 are given by elements of $T^{**}M$ which is larger than the (kinematic) tangent bundle TM. Thus in the case of Banach manifolds (even the ones having a global chart, as for instance Banach spaces), the notions of kinematic tangent vector and operational tangent vector do not coincide in general.

There are examples of Banach spaces possessing queer operational tangent vectors even in the reflexive case. A construction of homogeneous second order operational tangent vectors on Hilbert spaces was given in [KM97] and we explore it below for a class of Banach spaces. Let E be a Banach space and consider the natural inclusion of $E^* \times E^*$ into $L_2(E; \mathbb{R})$ by :

$$
\begin{array}{rcl}
E^* \times E^* & \to & L_2(E; \mathbb{R}) \\
(f, g) & \mapsto & (f \otimes g : (v, w) \mapsto f(v)g(w)) .
\end{array}
\tag{2.3}
$$

In general (contrary to the finite-dimensional case) the linear span of its image may not be dense. A functional $\ell \in (L_2(E; \mathbb{R}))^*$ defines an operational tangent vector of order 2 at any $a \in E$ by

$$
\delta_\ell f = \ell(f_a'')
\tag{2.4}
$$

if and only if it vanishes on $E^* \times E^*$ regarded as a subspace of $L_2(E; \mathbb{R})$ via (2.3). We also recall here that we can identify $L_2(E; \mathbb{R})$ with $L(E; E^*)$.

Proposition 2.4. *There are no operational tangent vectors of the second order on the Banach space l^p of p-summable sequences for $2 < p < \infty$. On the other hand, if $1 \le p \le 2$ there are non-trivial operational tangent vectors of the second order.*

Proof. The existence of operational tangent vectors of the second order follows from [KM97, Rem. 28.8]. Namely according to Pitt's theorem, every map from l^p to $(l^p)^*$ is compact if $2 < p < \infty$, see e.g. [Pit36], [Rya02, Thm. 4.23], [FHH$^+$01, Prop 6.25]. Moreover since all $(l^p)^*$ spaces have the approximation property, the closure of linear span of $(l^p)^* \times (l^p)^*$ coincides with the space of compact operators from l^p to $(l^p)^*$ [Rya02, Ch. 4]. So, the only functional ℓ which would vanish on $E^* \times E^*$ is the zero functional. Thus there are no non-zero operational tangent vectors of the second order on l^p for $2 < p < \infty$.

In the case $1 \leq p \leq 2$, the inclusion map $\iota : l^p \hookrightarrow (l^p)^*$ is not compact, so using Hahn–Banach theorem it is possible to define a non-zero functional ℓ on $L_2(E; \mathbb{R})$ that vanishes on the image of the map (2.3). This implies the existence of non-zero operational tangent vectors of the second order on l^p for $1 \leq p \leq 2$. $\qquad\square$

In particular for $p = 2$ we obtain an operational tangent vector of the second order on the separable Hilbert space $\mathcal{H}$. We will present this case more explicitly.

Example 2.5 (concrete queer operational vector on a Hilbert space). The Banach space $L_2(\mathcal{H}; \mathbb{R})$ can be identified with the Banach space of bounded operators $L^\infty(\mathcal{H})$. This identification maps a bilinear map B to the operator A defined by

$$B(v, w) = \langle Av, w \rangle \tag{2.5}$$

using Riesz theorem. The closure of the linear span of $\mathcal{H}^* \times \mathcal{H}^*$ considered as a subspace of $L_2(\mathcal{H}; \mathbb{R}) \simeq L^\infty(\mathcal{H})$ by inclusion (2.3) is the ideal of compact operators on $\mathcal{H}$. One can now obtain ℓ with required properties by putting e.g. $\ell(\mathbb{1}) = 1$ where $\mathbb{1}$ denotes the identity map, and $\ell(K) = 0$ for any compact operator $K \in L^\infty(\mathcal{H})$ and extend ℓ to the whole $L^\infty(\mathcal{H})$ by means of Hahn–Banach theorem.

Let us now demonstrate explicitly that the operational tangent vector δ_ℓ given by (2.4) with ℓ defined as above is not a kinematic tangent vector. Without loss of generality we fix the point $a = 0$. Taking for example the function

$$\rho(v) = \langle v, v \rangle \tag{2.6}$$

for $v \in \mathcal{H}$, we get $\rho''_v = 2\,\mathbb{1}$, where we have used the identification $L_2(\mathcal{H}; \mathbb{R}) \simeq L^\infty(\mathcal{H})$ given by (2.5). From definition it follows that $\delta_\ell(\rho) = 2$. On the other hand, any kinematic tangent vector to $\mathcal{H}$ at 0 can be identified with some $w \in \mathcal{H}$ and

$$w \cdot \rho = \langle w, 0 \rangle + \langle 0, w \rangle = 0.$$

Thus δ_ℓ is in fact a queer tangent vector. One can extend δ_ℓ to a queer constant operational vector field on $\mathcal{H}$, which we will denote by the same symbol.

Let us show that existence of δ_ℓ in Example 2.5 is not a contradiction with [AMR02, Thm. 4.2.16] stating that for manifolds M modelled on Banach spaces with norm smooth away from the origin, a certain space of derivations is isomorphic to the vector space of kinematic vector fields on M. In this reference, a derivation $\mathbf{D}$ on the Banach manifold M is a collection of linear maps $C^\infty(M, F) \to C^\infty(M, F)$ for all Banach spaces F, such that for any $f \in C^\infty(M, F)$, $g \in C^\infty(M, G)$, and any bilinear map $B : F \times G \to H$, the following Leibniz rule holds

$$\mathbf{D}\left(B(f, g)\right) = B(\mathbf{D}f, g) + B(f, \mathbf{D}g), \tag{2.7}$$

where F, G, and H are Banach spaces. An example of such a derivation is the Lie derivative. However the operational vector field δ_ℓ cannot be extended to a derivation in the sense of [AMR02].

Proposition 2.6. *The queer operational vector field δ_ℓ constructed in Example 2.5 cannot be extended to a derivation on all $C^\infty(\mathcal{H}, F)$ spaces, where F is any Banach space.*

Proof. Let us assume that there exists an extension $\mathbf{D}_\ell$ of δ_ℓ. Let B be the natural duality pairing between $\mathcal{H}^*$ and $\mathcal{H}$. Consider the maps $f : \mathcal{H} \to \mathcal{H}^*$, $v \mapsto \langle v, \cdot \rangle$ and g equal to the identity map on $\mathcal{H}$. Then $B(f, g)(v) = \langle v, v \rangle = \rho(v)$, and

$$\mathbf{D}_\ell\big(B(f, g)\big)(v) = \delta_\ell(\rho)(v) = \ell(2\,\mathbb{1}) = 2.$$

On the other hand,

$$B(\mathbf{D}_\ell f, g)(v) + B(f, \mathbf{D}_\ell g)(v) = B(\mathbf{D}_\ell f(v), v) + \langle v, \mathbf{D}_\ell g(v) \rangle.$$

This expression vanishes for $v = 0$, hence (2.7) cannot be satisfied for any extension of δ_ℓ. □

Proposition 2.7. *Let δ be an operational vector field of finite order on a manifold M. Then the set of points at which it is queer is open while the set of points at which it is kinematic is closed in M.*

Proof. Let n be the order of δ. The set of points at which δ is not queer is the intersection $\bigcap\limits_{k=2}^{n} \ell_k^{-1}(0)$ of level sets of zero sections of coefficients $\ell_k : M \to \bigsqcup\limits_{m \in M} (L_k(T_m M; \mathbb{R}))^*$ of δ. Since ℓ_k are continuous, the above intersection is a closed set.

The set of points at which δ is kinematic is $\bigcap\limits_{k=2}^{n} \ell_k^{-1}(0) \cap \ell_1^{-1}(TM)$, where we regard TM as a subbundle of $\bigsqcup\limits_{m \in M} (L_1(T_m M; \mathbb{R}))^* = T^{**}M$. It is straightforward to check that TM is a closed subset of $T^{**}M$ using local trivialization. □

3. Queer Poisson brackets

In this paper we will construct Poisson brackets which are localizable in the sense of the following definition:

Definition 3.1. A **Poisson bracket** on a manifold M is a bilinear operation $\{\,\cdot\,,\,\cdot\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

   (i) skew-symmetry: $\{f, g\} = -\{g, f\}$;
   (ii) Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$;
   (iii) Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$;

for all $f, g, h \in C^\infty(M)$.

A Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on M is called **localizable** if it has a **localization**, that is, a family consisting of a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_U$ on every open subset $U \subseteq M$, which satisfy $\{\,\cdot\,,\,\cdot\,\}_M = \{\,\cdot\,,\,\cdot\,\}$ and are compatible with restrictions, i.e., if $U \subseteq V$ and $f, g \in C^\infty(V)$ then $\{f, g\}_V|_U = \{f|_U, g|_U\}_U$. If this is the case, then for any function $h \in C^\infty(M)$, its corresponding **Hamiltonian vector field** is the operational vector field given by

$$X_h(f)(m) := \{h|_U, f\}_U(m) \tag{3.1}$$

for all $f \in C^\infty(U)$ and $m \in U$, for every open subset $U \subseteq M$.

Remark 3.2. A version of Peetre's theorem on a Banach space E was proved in [WD73], to the effect that if a linear map $T : C^\infty(E) \to C^\infty(E)$ is local in the sense that $\operatorname{supp} Tf \subset \operatorname{supp} f$ for all $f \in C^\infty(E)$, then T is a differential operator of locally finite order provided that E satisfies the condition of B^∞ smoothness (existence of bump functions with Lipschitz property for all derivatives). This condition is satisfied for Hilbert spaces, but e.g. not for the Banach space of real sequences that are convergent to zero.

From compatibility with restrictions it follows that operational vector fields (including Hamiltonian vector fields) are local in this sense. Thus in the case of B^∞ smooth Banach spaces they are differential operators of locally finite order.

In the following we denote by $\bigwedge^2 T^{**}M$ the bundle of skew-symmetric bilinear functions on the fibers of T^*M, for any Banach manifold M.

Definition 3.3. A Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on M is of **order one** at $m \in M$ if there exists a skew-symmetric bounded bilinear functional $\Pi_m \colon T_m^*M \times T_m^*M \to \mathbb{R}$ with

$$\{f, g\}(m) = \Pi_m(f_m', g_m') \tag{3.2}$$

for all $f, g \in C^\infty(M)$. Otherwise we say that $\{\,\cdot\,,\,\cdot\,\}$ is **queer** at $m \in M$.

If there exists a smooth section Π of the bundle $\bigwedge^2 T^{**}M$ satisfying (3.2) at every point $m \in M$, then we say that Π is the **Poisson tensor** of the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$.

Remark 3.4. In the above definition, if the Poisson bracket is of order one at some point $m \in M$ then it is not clear that there exists only one functional Π_m satisfying (3.2), as the differentials of globally defined functions at a given point m might not span the whole T_m^*M. However this is certainly the case if for instance Poisson bracket is localizable or M has a global chart.

Theorem 3.5. *Let δ_1 and δ_2 be two commuting operational vector fields on a Banach manifold M, and define*

$$\{f_1, f_2\}_U := (\delta_1)_U(f_1)\,(\delta_2)_U(f_2) - (\delta_2)_U(f_1)\,(\delta_1)_U(f_2),$$

for all f_1, $f_2 \in C^\infty(U)$, for every open subset $U \subseteq M$. Then $\{\,\cdot\,,\,\cdot\,\} := \{\,\cdot\,,\,\cdot\,\}_M$ is a localizable Poisson bracket with a localization consisting of the brackets $\{\,\cdot\,,\,\cdot\,\}_U$. If moreover δ_1 and δ_2 are linearly independent at some point $m \in M$, then the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ is queer at the point m if and only if at least one the operational vector field δ_1 and δ_2 is queer at m.

Proof. Bilinearity and skew-symmetry of $\{\,\cdot\,,\,\cdot\,\}$ are obvious. Jacobi identity follows from the commutativity of δ_1 and δ_2 just like in the case of canonical Poisson bracket on $\mathbb{R}^2$. This can also be seen e.g. as the special case $n = 2$ of [Fil85, Prop. 2]. The Leibniz rule for $\{\,\cdot\,,\,\cdot\,\}$ follows easily from (2.1). Compatibility with restrictions follows from the definition of operational vector fields.

Now assume that δ_1 and δ_2 are linearly independent at $m \in M$. If none of δ_1 and δ_2 is queer at m, then it follows by Remark 2.3 that their values at m satisfy $(\delta_1)_m, (\delta_2)_m \in T_m^{**}M$. Then (3.2) is satisfied if we define $\Pi_m \colon T_m^*M \times T_m^*M \to \mathbb{R}$ by

$$\Pi_m(\mu, \nu) = (\delta_1)_m(\mu)\,(\delta_2)_m(\nu) - (\delta_2)_m(\mu)\,(\delta_1)_m(\nu) \text{ for all } \mu, \nu \in T_m^*M,$$

hence $\{\,\cdot\,,\,\cdot\,\}$ is not queer at $m \in M$.

Conversely, assume that $\{\,\cdot\,,\,\cdot\,\}$ is not queer at $m \in M$, hence we have (3.2). Since the linear functionals $(\delta_1)_m, (\delta_2)_m \colon C_m^\infty(M) \to \mathbb{R}$ are linearly independent by hypothesis, there exists a germ $[f_1] \in C_m^\infty(M)$ with $\delta_1(f_1) = 0$ and $\delta_2(f_1) \neq 0$. Then for every $f \in C^\infty(U)$ we obtain $\{f_1, f\}_U(m) = \delta_2(f_1)(m)\,\delta_1(f)(m)$, hence by (3.2)

$$\delta_1(f)(m) = \frac{1}{\delta_2(f_1)(m)}\Pi_m((f_1)'_m, f'_m)$$

and this shows that the operational tangent vector $(\delta_1)_m$ has order 1 at m. One can similarly prove that the operational tangent vector $(\delta_2)_m$ has order 1 at m and this completes the proof. $\qquad\square$

One can use Theorem 3.5 and Proposition 2.4 to construct queer Poisson brackets on l^p spaces for $1 \leq p \leq 2$. Again we will present the case $p = 2$ in more detail.

Example 3.6 (concrete queer Poisson bracket). Now let us take $M = \mathcal{H} \times \mathbb{R}$. Denote points of M as (v, x). As the first operational vector field let us take δ_ℓ from Example 2.5 acting in v variable, and for the second $-\frac{\partial}{\partial x}$. They commute and thus by Theorem 3.5 define a queer Poisson bracket on $\mathcal{H} \times \mathbb{R}$:

$$\{f, g\}(v, x) := \delta_\ell(v) f(\cdot, x) \frac{\partial g}{\partial x}(v, x) - \frac{\partial f}{\partial x}(v, x) \delta_\ell(v) g(\cdot, x).$$

Note that this Poisson bracket has pathological properties: it does not allow Hamiltonian formalism in the usual sense since its corresponding Hamiltonian vector fields are in general only operational vector fields, e.g. for the function $h(v, x) = -x$ is

$$X_h := \{h, \cdot\} = \delta_\ell.$$

Thus it is not a section of TM. Since in the constructed example δ_ℓ was a differential operator of the second order, it will not lead to an evolution flow on M. Note that the system of Hamilton equations

$$\frac{d}{dt} f(v(t), x(t)) = (X_h f)(v(t), x(t))$$

for $f \in C^\infty(M)$ is not even a well posed problem. Namely for the function ρ given by (2.6) we get

$$\frac{d}{dt} \rho(v(t)) = 2. \tag{3.3}$$

Now consider the function $f(v, x) = \langle v, w \rangle$ for a fixed vector $w \in \mathcal{H}$. One sees that $f'' = 0$ and thus $X_h f = 0$. Since the vector w was arbitrary, it follows that $\frac{d}{dt} v(t) = 0$.

As demonstrated a queer Poisson bracket does not lead to the dynamics in the usual way. However it may be possible to consider the dynamics not on the initial manifold but on some jet bundle or higher (co)-tangent bundle, see e.g. [BGG15] and references therein.

Taking this into account, from the point of view of applications in physics (including classical mechanics) one should explicitly assume the existence of Poisson tensor in the definition of Poisson Banach manifold. This also ensures the existence of the map $\sharp : T^*M \to T^{**}M$ defined by

$$\sharp(\mu_m) = \Pi_m(\mu_m, \cdot), \qquad \mu_m \in T_m^*M. \tag{3.4}$$

Definition 3.7. A **Banach Poisson manifold** $(M, \{\,\cdot\,,\,\cdot\,\})$ is a Banach manifold M equipped with a localizable Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ for which there exists a Poisson tensor. A Banach Poisson manifold $(M, \{\,\cdot\,,\,\cdot\,\})$ is called **kinematic** if the corresponding $\sharp$ map satisfies

$$\sharp(T^*M) \subset TM. \tag{3.5}$$

This definition is a clarification of the definition of Banach Poisson manifolds given in [OR03, Def. 2.1], where the localizability property and the existence of Poisson tensor or $\sharp$ map were not explicitly assumed, but were assumed implicitly. In consequence all Banach Poisson manifolds considered there (including Banach Lie–Poisson spaces) do satisfy the corrected definition. Moreover, the Poisson tensor of every Banach Lie-Poisson space is uniquely determined. (See Remark 3.4.)

The condition (3.5) on the map $\sharp$ was introduced in [OR03] and guarantees that Hamiltonian vector fields are kinematic and it is equivalent to the bilinear functional $\Pi_m \colon T^*M \times T^*M \to \mathbb{R}$ being separately weak*-continuous.

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