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Special vector fields on Riemannian manifolds of constant negative sectional curvature and conservation laws

Keti Tenenblat and Alice Barbora Tumpach

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Abstract. We show that any n-dimensional Riemannian manifold with constant negative sectional curvature admits local orthonormal vector fields such that one of them v_1 is tangent to geodesics and the other $n-1$ vector fields are tangent to horocycles. We prove that the 1-form dual to v_1 is a closed form. We show how the closed form can be used to obtain conservation laws for PDEs whose generic solutions define metrics on open subsets with constant negative sectional curvature. These results extend to higher dimensions the 2-dimensional case proved in the 1980s. We prove that there exist local coordinates on the manifold such that the coordinate curves are tangent to the orthonormal vector fields. We apply the theory to obtain conservation laws for the Camassa-Holm equation ($n = 2$) and for the Intrinsic Generalized Sine-Gordon equation ($n \geq 2$).

Contents

1. Introduction

In [**4**],[**5**], the authors proved that any 2-dimensional Riemannian manifold, with constant negative Gaussian curvature, admits orthonormal vector fields v_1, v_2

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tangent to geodesics and horocycles respectively. In particular, they showed that the 1-form dual to v_1 is a closed form. The importance of the closed form is due to the fact that it provides conservation laws for partial differential equations (or system of equations) for real valued functions, whose generic solutions define metrics on open subset of the plane, whose Gaussian curvature is constant negative (w.l.o.g. -1). These are the so called differential equations that describe pseudo-spherical surfaces (**pss**). Many well know differential equations related to physical phenomena decribe **pss** such as Schrödinger equation, short-pulse equation, KdV, etc. Actually there are infinitely many such differential equations. The reader can find an extensive literature with classification results of such PDEs in [**5**]-[**13**] and references within. Explicit conservation laws have been recently obtained for example in [**10**] and [**11**] for some of these equations, by applying the results in [**4**],

In this paper, we generalize the results mentioned above to higher dimensions. More precisely, we show that any n-dimensional Riemannian manifold M^n, whose sectional curvature is constant -1, admits local orthonormal vector fields v_i, $i = 1, ..., n$, such v_1 is tangent to geodesics and v_i, $i \geq 2$ are tangent to horocycles. In particular, we show that the 1-form dual to v_1 is a closed form (Theorem 2.2). Moreover, we prove that there exist local coordinates on M such that the coordinate curves are tangent to the vectors of the orthonormal frame (Theorem 2.3). In Theorem 2.5, we show how to obtain the closed form for any Riemannian manifold, with constant sectional curvature -1. This closed form provides conservation laws for PDEs whose generic solutions define metrics on open subsets of $\mathbb{R}^n$ with constant negative sectional curvature. We apply the results in dimension $n = 2$ to obtain conservation laws for the Camassa-Holm equation [**2**]. For arbitrary dimensions $n \geq 2$, in Theorem 2.6 we get conservation laws for the Intrinsic Generalized Sine-Gordon equation (IGSGE). This is an n-dimensional generalization of the classical sine-Gordon equation, whose generic solutions define metrics on open subsets of $\mathbb{R}^n$, whose sectional curvature is -1 (see Example 2). The IGSGE was introduced by Beals-Tenenblat in [**1**] (see also Chapter V in [**14**]), as an intrinsic version of the Generalized sine-Gordon equation that corresponds to submanifolds $M^n \subset R^{2n-1}$ [**15**]. In higher dimensions, very few equations or systems of equations are known to be integrable in some sense. The IGSGE is an n-dimensional system of PDEs that has Bäcklund transformation, superposition formula, and it can also be solved by the inverse scattering method [**1**].

This paper is organized as follows: In Section 2 we state our main results, in Section 3 we prove Theorems 2.2, 2.3 and 2.5. In Section 4, we obtain conservation laws for the Camassa-Holm equation and in the higher dimensional context, we prove Theorem 4.1 that shows how to apply Theorems 2.2 and 2.5 in order to obtain conservation laws from the closed 1-form and then we prove Theorem 2.6 for the IGSGE.

2. Main Results

We consider an n-dimensional Riemannian manifold (M^n, g), with constant negative sectional curvature which, without loss of generality, we may consider to be -1. We first recall the 2-dimensional case, which shows that a Riemannian manifold (M^2, g), with constant negative Gaussian curvature admits special vector fields, that are tangent to geodesics and horocycles.

THEOREM 2.1. [**4**] [**5**] *Let M^2 be a $\mathcal{C}^\infty$ Riemannian surface. M has constant Gaussian curvature -1 if, and only if, given orthonormal vectors v_1^0, v_2^0 tangent to M at $p_0 \in M$, there exists an orthonormal frame field v_1, v_2, locally defined, such that $v_i(p_0) = v_i^0$, $i = 1, 2$ and the associated dual 1-forms $\theta_1\, \theta_2$ and connection form θ_{12} satisfy*

$$\theta_{12} + \theta_2 = 0. \tag{2.1}$$

In this case, θ_1 is a closed form.

We observe that the vector fields v_1 and v_2 are tangent to geodesics and to horocycles respectively. In fact, it follows from the fact that $dv_1 = -\theta_2 v_2$ and $dv_2 = \theta_2 v_1$. We prove a higher dimensional version of the theorem above, whenever the Riemannian manifold has constant negative sectional curvature.

THEOREM 2.2. *Let (M^n, g) be an n-dimensional Riemannian manifold. M has constant sectional curvature -1 if, and only if, given $v_1^0, ..., v_n^0$ orthonormal vectors tangent to M at $p_0 \in M$, there exists an orthonormal frame field $v_1, ..., v_n$, locally defined, such that $v_i(p_0) = v_i^0$, $i = 1, ..., n$ and the associated dual forms $\theta_1, ..., \theta_n$ and connection form θ_{ij} satisfy*

$$\theta_{1i} + \theta_i = 0, \qquad \forall\, i \geq 2, \tag{2.2}$$

$$\theta_{ij} = 0, \qquad \forall\, i \neq j,\ 2 \leq i, j \leq n. \tag{2.3}$$

In this case, θ_1 is a closed form. In particular, v_1 is tangent to geodesics and v_i, $i \geq 2$ are tangent to horocycles.

The existence of special frames on a Riemannian manifold with constant negative sectional curvature, as in Theorem 2.2, will enable us to show that one can locally parametrize the manifold with coordinates whose tangent vectors are in the direction of the frame.

THEOREM 2.3. *Let (M^n, g) be a Riemannian manifold of constant sectional curvature -1. Let $v_1, ..., v_n$ be an orthonormal fame field locally defined on M such that the dual forms $\theta_1, ..., \theta_n$ and the connection forms θ_{ij} satisfy* (2.1) *if $n = 2$ and* (2.2)-(2.3) *if $n > 2$. Then there exist local coordinates $y_1, ..., y_n$ such that $\partial/\partial y_1 = v_1$ and $\partial/\partial y_i = r_i v_i$, $i \geq 2$, where r_i is a funcion of y_1 only.*

As an immediate corollary of Theorems 2.2 and 2.3 we have

THEOREM 2.4. *Let (M^n, g) be a Riemannian manifold of constant sectional curvature -1. Then given orthonormal vectors $v_1^0, ..., v_n^0$ at a point $p^0 \in M$, there exist local orthogonal coordinates $y_1, ..., y_n$ such that the curves that are tangent to $\partial/\partial y_1$ are geodesics and the curves tangent to $\partial/\partial y_i$, $i \geq 2$, are horocycles and at p^0 they are tangent to $v_1^0, ..., v_n^0$.*

Whenever $n = 2$, Theorem 2.1 has been applied to obtain conservation laws for differential equations that describe pseudo-spherical surfaces, i.e., differential equations or system of equations for real valued functions defined on open subsets of the plane, whose generic solutions define metrics with Gaussian curvature -1.

In order to consider such an application for higher dimensions, i.e., for systems of differential equations whose generic solutions define metric on open subsets of $\mathbb{R}^n$ with sectional curvature -1, we state our next result, that shows how to obtain the closed form given by Theorem 2.2.

Theorem 2.5. *Let* $(U \subset \mathbb{R}^n, g)$ *be an open subset* U *with a Riemannian metric* g*, whose sectional curvature is constant* -1 *and let* $e_1, ..., e_n$ *be any orthonormal frame with dual and connection forms* ω_i *and* ω_{ij} *respectively. Then there exists a unique orthonormal frame* $v_i = L_{ij} e_j$*,* $L(x) \in O(n)$*, for a given initial condition* $L(x^0) \in O(n)$*,* $x^0 \in U$*, such that* L *satisfies the integrable system of PDEs*

$$(dL\,L^t)_{1i} + (LWL^t)_{1i} + \sum_{k=1}^{n} L_{ik}\omega_k = 0, \qquad \forall i \geq 2, \tag{2.4}$$

$$(dL\,L^t)_{ij} + (LWL^t)_{ij} = 0, \qquad \forall\, i, j \geq 2,\, i \neq j, \tag{2.5}$$

where $(W)_{ij} = \omega_{ij}$*. In this case,* $\sum_{k=1}^{n} L_{1k}\omega_k$ *is a closed form.*

The existence of a closed form in higher dimensions provides conservation laws, as one can see in Theorem 4.1, when we fix one of the independent variables to be the time variable. As an important application, we consider the Intrinsic Generalized sine-Gordon equation, which is a system of differential equations for a pair $\{V(x), h(x)\}$ defined on an open set $x \in U \subset \mathbb{R}^n$, $n \geq 2$, where $V(x) = (V_1(x), ..., V_n(x))$ is a unit vector field and $h_{ij}(x)$ is an off diagonal $n \times n$ matrix valued function satisfying (4.13) (see Example 2). This equation reduces to the sine-Gordon equation when $n = 2$. By applying Theorem 2.5, we prove the following result.

Theorem 2.6. *The Intrinsic Generalized sine-Gordon equation admits at least* $n-1$ *conservation laws, considering one of the independent variables to be the time variable.*

3. Proof of the Main Results

In this section, we prove some of the results stated in Section 2. In order to do so, we need the following basic facts. Let (M^n, g) be a Riemannian manifold of constant sectional curvature K. Consider a local orthonormal frame field $e_1, ..., e_n$. Let $\omega_1, ..., \omega_n$ be its dual coframe and let $\omega_{ij} = -\omega_{ji}$ be the connection forms. Then the structure equations for M are

$$d\omega_i = \sum_{j \neq i,\, j=1}^{n} \omega_j \wedge \omega_{ji}, \tag{3.1}$$

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} \tag{3.2}$$

where the curvature $\Omega_{ij} = -K\omega_i \wedge \omega_j$ characterizes the fact that the sectional curvature is constant K.

In what follows, we will use the notion of *vector valued differential forms* on a manifold. In particular, let $v : M^n \to TM$ be a vector field on M, where TM is the tangent bundle of M. One can consider $v_p = \sum_{i=1}^{n} v^i(p)\, e_i(p)$, where $e_i(p)$ is a basis of the tangent space T_pM. Then $dv : TM \to TM$ is a vector valued differential form given by $dv(X) = \sum_{i=1}^{n} dv^i(X)\, e_i$, where dv^i is a 1-form and X is any tangent vector field on M. Just as in the case of ordinary differential forms, one can define operations on vector valued forms such as addition, multiplication by a function, wedge product and exterior derivatives acting component-wise relative to any basis of the vector space. In particular, denoting the metric by $\langle \cdot, \cdot \rangle$, whenever two vector fields v_i and v_j are such that $\langle v_i, v_j\rangle$ is constant, then $\langle dv_i(v_k), v_j\rangle + \langle v_i, dv_j(v_k)\rangle = 0$, for any vector field v_k.

Proof of Theorem 2.2. We have to prove that the system of equations (3.1) and (3.2) is integrable if, and only if, the sectional curvature K of M is constant -1. In order to do so, we will use Cartan-Kähler theory on exterior differentiable systems [**3**].

Let $\mathcal{I}$ be the ideal generated by $\gamma_i = \theta_{1i} + \theta_i$ and $\beta_{ij} = \theta_{ij}$, $i \neq j$, $i, j \geq 2$. Then, it follows from (3.1) and (3.2), that

$$\begin{aligned} d\gamma_i = d\theta_{1i} + d\theta_i &= d\theta_{1i} + \sum_{k=1}^{n} \theta_k \wedge \theta_{ki} \\ &= d\theta_{1i} + \theta_1 \wedge \theta_{1i} + \sum_{k=2}^{n} \theta_k \wedge \theta_{ki} \\ &= d\theta_{1i} + \theta_1 \wedge (\gamma_i - \theta_i) + \sum_{k=2}^{n} (\gamma_k - \theta_{1k}) \wedge \theta_{ki} \\ &= d\theta_{1i} - \sum_{k=2}^{n} \theta_{1k} \wedge \theta_{ki} - \theta_1 \wedge \theta_i \quad (\text{ mod } \mathcal{I}). \end{aligned} \tag{3.3}$$

Similarly, for $i, j \geq 2$

$$\begin{aligned} d\beta_{ij} = d\theta_{ij} &= \sum_{k=1}^{n} \theta_{ik} \wedge \theta_{kj} + \Omega_{ij} \\ &= \theta_{i1} \wedge \theta_{1j} + \sum_{k=2}^{n} \theta_{ik} \wedge \theta_{kj} + \Omega_{ij} \\ &= -(\gamma_i - \theta_i) \wedge (\gamma_j - \theta_j) + \sum_{k=2}^{n} \beta_{ik} \wedge \beta_{kj} + \Omega_{ij} \\ &= -\theta_i \wedge \theta_j + \Omega_{ij} \quad (\text{ mod } \mathcal{I}). \end{aligned} \tag{3.4}$$

Therefore, $d\beta_{ij} = 0 \text{ mod } \mathcal{I}$ whenever $\Omega_{ij} = \theta_i \wedge \theta_j$, for all $i, j \geq 2$. Hence, it follows from (3.3) and (3.4) that $\mathcal{I}$ is closed under exterior differentiation if and only if M has constant sectional curvature $K = -1$. The first part of the theorem follows from Frobenius theorem.

We now prove that θ_1 is a closed form. In fact, it follows from (2.2) and the structure equation (3.1) that

$$d\theta_1 = -\sum_{k=2}^{n} \theta_k \wedge \theta_{1k} = \sum_{k=2}^{n} \theta_k \wedge \theta_k = 0.$$

We observe that as a consequence of (2.2) and (2.3) we have $dv_1 = -\sum_{i=2}^{n} \theta_i v_i$ and $dv_i = \theta_i v_1$. Hence, $dv_1(v_1) = 0$ and $dv_i(v_i) = v_1$ Therefore, the vector fields $v_1, ..., v_n$ have the following property: v_1 is tangent to geodesics and v_i, $i \geq 2$ are tangent to horocycles.

□

Our next proof shows that manifolds of constant negative sectional curvature admit special coordinate systems locally defined as in Theorem 2.3.

Proof of Theorem 2.3. We start proving the 2-dimensional case, i.e., let M^2 be a Riemannian surface. Let v_1, v_2 be an orthonormal frame field locally defined on M such that the dual forms θ_1, θ_2 and the connection form θ_{12} satisfy (2.1). It follows from Theorem 2.1 that θ_1 is a closed form. Therefore, there exists a function G locally defined such that

$$dG = \theta_1. \tag{3.5}$$

We need to show that there exits a function r such that $v_2(r) = 0$ and the vector fields v_1 and rv_2 commute, i.e.,

$$dr(v_2) = 0 \qquad \text{and} \qquad [v_1,\, rv_2] = 0. \tag{3.6}$$

Since $\theta_{12} = \langle dv_1, v_2\rangle$ it follows from (2.1) that

$$\begin{array}{llll} \langle dv_1(v_1), v_2\rangle & = \theta_{12}(v_1) & = -\theta_2(v_1) = & 0, \\ \langle dv_1(v_2), v_2\rangle & = \theta_{12}(v_2) & = -\theta_2(v_2) = & -1. \end{array} \tag{3.7}$$

The second equality of (3.6) is equivalent to

$$dr(v_1)v_2 + rdv_2(v_1) - rdv_1(v_2) = 0.$$

This is a vector field that vanishes when its inner product with the basis vanishes. Taking inner products of this expression with v_1 and v_2, we get from (3.7) respectively

$$\begin{array}{l} r\langle dv_2(v_1), v_1\rangle - r\langle dv_1(v_2), v_1\rangle = -r\langle v_2, dv_1(v_1)\rangle = 0, \\ dr(v_1) - r\langle dv_1(v_2), v_2\rangle = dr(v_1) + r = 0. \end{array} \tag{3.8}$$

Therefore, $d(\log r)(v_1) = -1$. Moreover, from the first equality of (3.6) we have $d(\log r)(v_2) = 0$. Since $d(\log r) = d(\log r)(v_1)\theta_1 + d(\log r)(v_2)\theta_2$, we conclude from (3.8) that $d(\log r) = -\theta_1$. It follows from (3.5) that

$$r = c\exp(-G),$$

for some constant $c > 0$, i.e., the function r exists satisfying (3.6). Therefore, there exit coordinates y_1 and y_2 locally defined such that $\partial/\partial y_1 = v_1$ and $\partial/\partial y_2 = rv_2$, where r is a funcion of y_1 only.

We now prove the n-dimensional case. Let $v_1, ..., v_n$ be an orthonormal frame field locally defined on (M^n, g) such that the dual forms $\theta_1, ..., \theta_n$ and the connection forms θ_{ij} satisfy (2.2) and (2.3). It follows from Theorem 2.2 that θ_1 is a closed form. Therefore, there exists a function G locally defined such that

$$dG = \theta_1. \tag{3.9}$$

We need to show that there exit functions $r_j \neq 0$, $j \geq 2$, such $v_i(r_j) = 0$, for $i, j \geq 2$ and the vector fields $v_1, r_2v_2, ..., r_nv_n$ commute, i.e.,

$$dr_i(v_j) = 0 \qquad [v_1,\, r_iv_i] = 0 \quad \text{and} \quad [r_iv_i, r_jv_j] = 0, \quad \forall i, j \geq 2. \tag{3.10}$$

Since

$$\theta_{1i} = \langle dv_1, v_i\rangle \text{ and } \quad \theta_{ij} = \langle dv_i, v_j\rangle, \tag{3.11}$$

it follows from (3.11), (2.2) and (2.3) that for all $i \neq j$, $i, j \geq 2$ we have

$$\begin{array}{llll} \langle dv_1(v_1), v_i\rangle & = \theta_{1i}(v_1) & = -\theta_i(v_1) = & 0, \\ \langle dv_1(v_i), v_i\rangle & = \theta_{1i}(v_i) & = -\theta_i(v_i) = & -1, \\ \langle dv_1(v_i), v_j\rangle & = \theta_{1j}(v_i) & = -\theta_i(v_j) = & 0, \\ \langle dv_i(v_k), v_j\rangle & = \theta_{ij}(v_k) & = 0, \ \forall k. & \end{array} \tag{3.12}$$

The second and third equalities of (3.10) are equivalent to

$$dr_i(v_1)v_i + r_i dv_i(v_1) - r_i dv_1(v_i) = 0, \tag{3.13}$$

$$r_i dr_j(v_i)v_j + r_i r_j dv_j(v_i) - r_j dr_i(v_j)v_i - r_j r_i dv_i(v_j) = 0, \quad i, j \geq 2. \tag{3.14}$$

The expressions given by (3.13) and (3.14)are vector fields that vanish whenever the inner product with all vectors $v_1, ..., v_n$ vanish.

Taking inner product of (3.13) with v_1 and $v_j,\ j \geq 2$, from (3.12) and (3.11) and the fact that $\langle dv_1(v_i), v_1\rangle = 0$, we get that

$$\begin{cases} r_i\langle dv_i(v_1), v_1\rangle = -r_i\langle dv_1(v_1), v_i\rangle = 0, \\ dr_i(v_1)\delta_{ij} + r_i\delta_{ij} = 0, \end{cases}$$

respectively. Therefore, $[v_1,\ r_i v_i] = 0$ if and only if

$$d(\log r_i)(v_1) = -1. \tag{3.15}$$

Similarly, taking inner product of (3.14) with v_1 and $v_k,\ k \geq 2$, we get from (3.11) and the third equation of (3.12) that, for all $i, j \geq 2$,

$$\begin{cases} r_i r_j\,(\,\langle dv_j(v_i), v_1\rangle - \langle dv_i(v_j), v_1\rangle\,) \\ \qquad = r_i r_j(-\langle dv_1(v_i), v_j\rangle + \langle dv_1(v_j), v_i\rangle\,) = 0, \\ r_i dr_j(v_i)\delta_{jk} - r_j dr_i(v_j)\delta_{ik} = 0. \end{cases}$$

Therefore, $dr_k(v_i) = 0$ for $i \neq k,\ i, k \geq 2$. Now, since we also want the first equality of (3.10) to be satisfied, i.e., $dr_k(v_k) = 0$, we conclude that $[r_i v_i, r_j v_j] = 0$ whenever $dr_k(v_i) = 0$ for all $i, k \geq 2$. Since $d(\log r_i) = d(\log r_i)(v_1)\theta_1 + \sum_{j=2}^n d(\log r_i)(v_j)\theta_j$, we conclude from (3.15) that $d(\log r_i) = -\theta_1$. Therefore, it follows from (3.9) that

$$r_i = c_i \exp(-G),$$

for some constant $c_i > 0$, i.e., the functions r_i exist satisfying (3.10). Therefore, there exist coordinates $y_1, ...y_n$ locally defined such that $\partial/\partial y_1 = v_1$ and $\partial/\partial y_i = r_i v_i$, where r_i is a funcion of y_1 only.

□

In order to prove Theorem 2.5 we will use Theorem 2.2 that says that there are special vector fields on a Riemannian manifold of constant negative sectional curvature. Therefore we need the following basic result that shows how the dual and connection forms are affected under a change of orthonormal frame fields, on any Riemannian manifold.

LEMMA 3.1. *Let $(U \subset \mathbb{R}^n, g)$ be an open subset U with a Riemannian metric g, and $e_1, ..., e_n$ is an orthonormal frame on U with dual forms $\omega_1, ..., \omega_n$ and connection forms $\omega_{ij} = -\omega_{ji}$. If $v_1, ..., v_n$ is another orthonormal frame given by $v_i = \sum_{j=1}^n L_{ij}e_j$ where $L(x) \in O(n)$ $x \in U$, then its dual forms θ_i and its connection forms θ_{ij} are given by*

$$\theta_i = \sum_{j=1}^n L_{ij}\omega_j \quad \text{and} \quad \theta_{ij} = (dL\,L^t)_{ij} + (LWL^t)_{ij}, \tag{3.16}$$

where L^t is the transpose of L and $(W)_{ij} = \omega_{ij}$.

The proof is a straightforward computation showing that $\theta_i(v_\ell) = \delta_{i\ell}$ and $\theta_{ij} = \langle dv_i, v_j\rangle$.

Proof of Theorem 2.5. The proof of Theorem 2.5 follows from Theorem 2.2 and Lemma 3.1.

□

4. Applications

In this section, we provide some applications of the results stated in Section 2.

4.1. The two dimensional case. Theorem 2.1 has been applied in order to obtain conservation laws for several PDEs for a real function $u(x,t)$ (or systems of PDEs) that describe pseudo-spherical surfaces (see [**4**], [**5**]). Such an equation is characterized by the fact that its generic solutions define metrics, on open subsets of the plane, whose Gaussian curvature is constant -1, i.e., they define 1-forms ω_1 and ω_2 and the connection forms ω_{12} in terms of $u(x,t)$ and its derivatives which satisfy the structure equations

$$\begin{aligned} d\omega_1 &= \omega_2 \wedge \omega_{21} \\ d\omega_2 &= \omega_1 \wedge \omega_{12} \\ d\omega_{12} &= \omega_1 \wedge \omega_2 \end{aligned} \tag{4.1}$$

The metric is defined by $ds^2 = \omega_1^2 + \omega_2^2$.

The conservation laws for the differential equations are obtained as a consequence of Theorem 2.1 as follows. Consider the 1-forms ω_i, $i = 1,2$ and ω_{12} satisfying (4.10). Then there are orthonormal vector fields e_1 and e_2 whose dual forms are ω_1 and ω_2. Moreover, Theorem 2.1 says that there exist special orthonormal frames v_1 and v_2 such that its dual forms θ_1, θ_2 and its connection form θ_{12} satisfy $\theta_{12} + \theta_2 = 0$, and in this case θ_1 is a closed form.

Observe that the frames e_1, e_2 and v_1, v_2 are related by an angle function ϕ, namely

$$\begin{aligned} v_1 &= \cos\phi\, e_1 - \sin\phi\, e_2, \\ v_2 &= \sin\phi\, e_1 + \cos\phi\, e_2, \end{aligned}$$

and hence the dual forms and the connection forms are related by

$$\begin{aligned} \theta_1 &= \cos\phi\, \omega_1 - \sin\phi\, \omega_2, \\ \theta_2 &= \sin\phi\, \omega_1 + \cos\phi\, \omega_2, \\ \theta_{12} &= \omega_{12} - d\phi. \end{aligned} \tag{4.2}$$

Therefore, it follows from (2.1) that ϕ is determined up to constants by the following equation

$$d\phi = \omega_{12} + \sin\phi\, \omega_1 + \cos\phi\, \omega_2, \tag{4.3}$$

and the closed 1-form θ_1 is given in terms of ϕ by (4.2), i.e.,

$$\theta_1 = \cos\phi\, \omega_1 - \sin\phi\, \omega_2. \tag{4.4}$$

Assume that, in a coordinate system $(x,t) \in U \subset R^2$, the 1-forms are given as follows

$$\omega_i = f_{i1}\, dx + f_{i2}\, dt \qquad \omega_{12} = f_{31}\, dx + f_{32}\, dt,$$

where $i = 1,2$ and $f_{ij}(x,t)$ are differentiable functions. Then the angle function $\phi(x,t)$ is determined in terms of f_{ij} by (4.3), i.e., by the completely integrable system of equations

$$\begin{cases} \phi_x = f_{31} + f_{11}\sin\phi + f_{21}\cos\phi, \\ \phi_t = f_{32} + f_{12}\sin\phi + f_{22}\cos\phi. \end{cases} \tag{4.5}$$

Moreover, if $\phi(x,t)$ is any solution of (4.5), then (4.4) implies that

$$(f_{11}\cos\phi - f_{21}\sin\phi)\,dx + (f_{12}\cos\phi - f_{22}\sin\phi)\,dt \tag{4.6}$$

is a closed form that provides a conservation law.

Assume that moreover the functions f_{ij} are analytic in a parameter η, then the angle function ϕ and the closed form (4.6) will also be analytic in η. In this case, we consider $\phi = \sum_{j=0}^{\infty}\phi_j\eta^j$. Then the Laurent expansion of (4.5) will provide the functions ϕ_j, $j \geq 0$ and the expansion of (4.6) will provide infinitely many closed forms, by considering each coefficient of the powers of η. When one considers solutions that are periodic on x or solutions that decay to zero when $x \to \pm\infty$, then one gets infinitely many conserved quantities.

We mention recent results in [**10**] and [**11**], where the conservation laws were explicitly given for a Pholmeyer-Lund Regge type system of equations and for a vector Short pulse equation. In this paper, we apply this procedure in detail to the Camassa-Holm equation.

Example 1. Consider the Camassa-Holm equation

$$u_t - u_{xxt} = uu_{xxx} + 2u_xu_{xx} - 3uu_x - mu_x. \tag{4.7}$$

whose generic solutions define metrics on open sets of the plane (x,t), whose Gaussian curvature is constant -1. In fact, in [**13**] it was shown that considering the 1-forms

$$\begin{aligned} \omega_i &= f_{i1}dx + f_{i2}dt, \qquad i = 1,2,\\ \omega_{12} &= f_{31}dx + f_{32}dt, \end{aligned}$$

where

$$\begin{array}{ll} f_{11} = h - 1 + \frac{\eta^2}{2} & f_{12} = -u(f_{11}+1) + \eta u_x - \frac{m}{2} - \frac{\eta^2}{2} + 1,\\ f_{21} = \eta & f_{22} = u\eta + u_x - \eta,\\ f_{31} = h + \frac{\eta^2}{2} & f_{32} = -uf_{31} + \eta u_x - u - \frac{m}{2} - \frac{\eta^2}{2}, \end{array}$$

and $h(x,t) = u - u_{xx} + m/2$, the metric defined by $ds^2 = \omega_1^2 + \omega_2^2$, where ω_{12} is the connection form, has Gaussian curvature -1 . More precisely, the structure equations (4.1) are satisfied if, and only if, the function $u(x,t)$ satisfies (4.7).

Since the functions f_{ij} are analytic in η, it follows that the angle function ϕ satisfying (4.5) is also analytic in η and we can consider $\phi = \sum_{j=0}^{\infty}\phi_j\eta^j$. Therefore, the closed 1-form (4.6) is also analytic in η. By considering each coefficient of η in (4.5), we get the integrable system of differential equations that provide the functions ϕ_j, $j \geq 0$ and the Laurent expansion of (4.6) provides infinitely many closed forms considering the coefficients of the powers of η.

More precisely, from the Laurent expansion of (4.5), considering the coefficient independent of η, the following system is integrable for ϕ_0 (i.e. the mixed derivatives commute), for any solution u of (4.7).

$$\begin{cases} \phi_{0,x} = (h-1)\sin\phi_0 + h,\\ \phi_{0,t} = \cos\phi_0 u_x - \sin\phi_0(uh + \frac{m}{2} - 1) - u(h+1) - \frac{m}{2}. \end{cases}$$

Therefore, there exists a unique ϕ_0 for a given initial condition $\phi_0(x_0,t_0)$. Moreover, from (4.6) we have a closed form for (4.7) given by

$$\cos\phi_0(h-1)\,dx + [\cos\phi_0(-uh + 1 - m/2) - u_x\sin\phi_0]\,dt. \tag{4.8}$$

Similarly, considering u and ϕ_0 as above, the following system is integrable for ϕ_1

$$\begin{cases} \phi_{1,x} = [(h-1)\phi_1 + 1]\cos\phi_0, \\ \phi_{1,t} = -[(uh - 1 + m/2)\cos\phi_0 + u_x \sin\phi_0]\phi_1 - (u+1)\cos\phi_0 + u_x(\sin\phi_0 + 1), \end{cases}$$

and we have a second closed form for (4.7) given by

$$-\{[(h-1)\phi_1+1]\sin\phi_0\}\,dx + \{u_x(\phi_1+1)\cos\phi_0 + [(uh+m/2-1)\phi_1+u+1]\sin\phi_0\}\,dt. \tag{4.9}$$

We observe that this procedure goes on for all coefficients of η in the Laurent expansion, providing infinitely many closed forms for the Camassa Holm equation and hence conserved quantities in time when the functions are periodic in x or decay appropriately when $x \to \infty$.

4.2. Higher dimensional case. We will now show how Theorems 2.2 and 2.5 can be applied in the higher dimensional context. We consider a system of PDEs for functions with n independent variables $(x_1, ..., x_n)$, $n \geq 2$, whose generic solutions define Riemannian metrics on open sets of $\mathbb{R}^n$ such that the sectional curvature is constant -1. This means that we have 1-forms $\omega_1, ..., \omega_n$ and connection 1-forms $\omega_{ij} = -\omega_{ji}$, $1 \leq i, j \leq n$ given in terms of the solutions of the PDEs and its derivatives, such that the following structure of equations are satisfied

$$\begin{aligned} d\omega_i &= \textstyle\sum_{j\neq i,\, j=1}^{n} \omega_j \wedge \omega_{ji}, \\ d\omega_{ij} &= \textstyle\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \omega_i \wedge \omega_j. \end{aligned} \tag{4.10}$$

The metric is defined by $ds^2 = \sum_i^n \omega_i^2$, where ω_i $i = 1, ...n$, are linearly independent.

The following result shows how to obtain conservation laws from a closed 1-form in higher dimensions.

THEOREM 4.1. *Consider local coordinates $x = (x_1, x_2, ..., x_n) \in U \subset \mathbb{R}^n$, $n > 2$ and a 1-form $\theta = \sum_{j=1}^n f_j(x)dx_j$, where f_j are differentiable functions of x. Assume that x_1 is the time variable that is denoted by t. If θ is a closed form then for each $j \geq 2$, the $(n-1)-$forms*

$$\begin{aligned} \psi_2 &= \theta \wedge dx_3 \wedge dx_4 \wedge \ldots \wedge dx_n, \\ \psi_3 &= \theta \wedge dx_2 \wedge dx_4 \wedge \ldots \wedge dx_n, \\ &\vdots \\ \psi_n &= \theta \wedge dx_2 \wedge \ldots \wedge dx_{n-1} \end{aligned} \tag{4.11}$$

are conservation laws and hence

$$\frac{\partial f_1}{\partial x_j} - \frac{\partial f_j}{\partial t} = 0, \qquad \forall j \geq 2. \tag{4.12}$$

Moreover, if the functions f_j are analytic in a parameter η, then the closed form may provide infinitely many conservation laws.

Proof. Since $\theta = f_1(x)\,dt + \sum_{\ell=2}^{n} f_\ell(x)dx_\ell$, is a closed form, it follows that for all $j \geq 2$, ψ_j is also a closed form.

$$\begin{aligned} d\psi_j &= d\theta \wedge dx_2 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots dx_n \\ &= \sum_{\ell=2}^{n} \left(\frac{\partial f_1}{\partial x_\ell} dx_\ell \wedge dt + \frac{\partial f_j}{\partial x_\ell} dx_\ell \wedge dx_j \right) \wedge dx_2 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots dx_n \\ &= \left(\frac{\partial f_j}{\partial t} - \frac{\partial f_1}{\partial x_j} \right) dt \wedge dx_2 \wedge \ldots \wedge dx_n. \end{aligned}$$

Since $d\psi_j = 0$ and dt, $dx_2, \ldots dx_n$ are linearly independent we get that (4.12) holds.

If the functions f_j are analytic in a parameter η, then each coefficient of the Taylor expansion of θ in terms of η, provides a closed form and hence a conservation law.

□

Remarks:
1. Theorem 4.1 shows that for each $j \geq 2$, $\int f_j dx_j$ is a conserved quantity in time, for functions f_1 that decay appropriately when $x_i \to \infty$ for $i = 2, ..., s$ for some $s \leq n-1$ and f_1 is periodic on the remaining variables x_j, $j = s+1, ..., n$, i.e., f_1 is defined on $R \times R^s \times T^{n-s-1}$. In fact, it follows from (4.12) that

$$\frac{\partial}{\partial t} \int f_j dx_j = \int \frac{\partial f_1}{\partial x_j} dx_j = 0,$$

i.e. (4.12) provides $n-1$ conserved quantities.
2. The closed form θ of Theorem 4.1, besides implying (4.12), it also shows that for $i, j \geq 2$ and $i < j$, the functions f_i and f_j satisfy the following relations

$$\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = 0.$$

As an application of Theorem 2.5, we will show how to obtain conservation laws for the Intrinsic Generalized sine-Gordon (the sine-Gordon equation when $n = 2$).

Example 2. The *Intrinsic Generalized sine-Gordon* (IGSGE) is a system of second order differential equations for a unit vector field $V(x) = (V_1(x), V_2(x), ..., V_n(x))$, $x \in U \subset \mathbb{R}^n$ that can be regarded as a first order system of equations for the pair $\{V, h\}$, where h is an off-diagonal $(n \times n)$-matrix valued function determined by the first derivatives of V (when V_i do not vanish), see the second equation in (4.13)), given by

$$\begin{aligned} &VV^t = 1, \\ &\frac{\partial V_i}{\partial x_j} = V_j h_{ji}, \\ &\frac{\partial h_{ij}}{\partial x_i} + \frac{\partial h_{ji}}{\partial x_j} + \sum_{s \neq i,j} h_{si} h_{sj} = V_i V_j, \qquad i \neq j, \\ &\frac{\partial h_{ij}}{\partial x_s} = h_{is} h_{sj}, \qquad i, j, s \text{ distinct}. \end{aligned} \tag{4.13}$$

where $1 \leq i, j, s \leq n$. When $n = 2$, by taking $V = (\cos u/2, \sin u/2)$, $u(x_1, x_2)$, the system (4.13) reduces to the sine-Gordon equation

$$u_{x_1x_1} - u_{x_2x_2} = \sin u.$$

Explicit solutions of the IGSGE are given for example by

$$V_1 = \tanh x_1, \qquad V_j = c_j \frac{1}{\cosh x_1}, \qquad \text{where } x_1 > 0, \text{ and } \sum_{j=2}^{n} c_j^2 = 1.$$

Other solutions for (4.13) can be found in [**14**] (Example b) page 142 and Proposition 3.1 on page 143).

We observe that considering a pair $\{V, h\}$ satisfying (4.13) such that V_i do not vanish on an open subset $U \subset \mathbb{R}^n$, then the unit vector field V defines a Riemannian metric on U with constant sectional curvature -1. In fact, considering the one forms

$$\omega_i = V_i dx_i, \qquad \omega_{ij} = h_{ij} dx_j - h_{ji} dx_i, \tag{4.14}$$

the metric is defined by $ds^2 = \sum_{i=1}^{n} \omega_i^2$ and the 1-forms ω_i and the connection forms ω_{ij} satisfy the structure equations (4.10), as a consequence of (4.13). Observe that the third and fourth equations of the system (4.13) are the Gauss equation of the metric.

Proof of Theorem 2.6 . As we have seen above, any solution $\{V, h\}$ of the IGSGE defines a metric on an open subset $U \subset \mathbb{R}^n$, whose sectional curvature is constant -1 by considering the 1-forms ω_i and ω_{ij} defined by (4.14).

Let e_i, $i = 1, ..., n$ be the orthonormal vector fields whose dual forms are ω_i. It follows from Theorem 2.5, that there exists a unique orthonormal frame $v_i = L_{ij} e_j$, $L(x) \in O(n)$, for a given initial condition $L(x^0)$, $x^0 \in U$ such that for all $i, j \geq 2$, $i \neq j$, equations (2.4) and (2.5) hold and in this case $\sum_{k=1}^{n} L_{1k} \omega_k$ is a closed form. Hence, considering (4.14), the system of equations (2.4) and (2.5) reduces to

$$(dL\, L^t)_{1i} + (LWL^t)_{1i} + \sum_{k=1}^{n} L_{ik} V_k \, dx_k = 0, \tag{4.15}$$

$$(dL\, L^t)_{ij} + (LWL^t)_{ij} = 0, \tag{4.16}$$

where $i, j \geq 2$, $i \neq j$ and

$$W_{ij} = \omega_{ij} = h_{ij} dx_j - h_{ji} dx_i. \tag{4.17}$$

In this case,

$$\theta = \sum_{k=1}^{n} L_{1k} V_k dx_k \tag{4.18}$$

is a closed form. Hence, it follows from Theorem 4.1 that, assuming that $x_1 = t$ is a time variable, then for all $k \geq 2$, the $(n-1)-$forms ψ_k given by (4.11) are conservation laws. and hence

$$\frac{\partial (L_{1k} V_k)}{\partial t} - \frac{\partial (L_{11} V_1)}{\partial x_k} = 0. \tag{4.19}$$

□

As we mentioned in Example 2, whenever $n = 2$, the Intrinsic Generalized Sine-Gordon Equation reduces to the classical sine-Gordon equation $u_{x_1x_1} - u_{x_2x_2} = \sin u$, for a real valued function $u(x_1, x_2)$, when we consider $V = (\cos(u/2), \sin(u/2))$. We want to exhibit the conservation law given by Theorem 2.6 for the sine-Gordon equation.

In this case, for any solution $u(x_1, x_2)$ of the sine-Gordon equation, it follows from the second equation (4.13) that $h_{12} = u_{x_1}/2$ and $h_{21} = -u_{x_2}/2$. Therefore, (4.14) provides the metric and the connection form as

$$\omega_1 = \cos\frac{u}{2}dx_1, \qquad \omega_2 = \sin\frac{u}{2}, \qquad \omega_{12} = \frac{u_{x_1}}{2}dx_2 + \frac{u_{x_2}}{2}dx_1.$$

It follows from Theorem 2.5 that there exists an orthogonal matrix

$$L = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

where $\phi(x_1, x_2)$, that satisfies (4.16) and (4.15). Then, (4.16) cannot occur since $n = 2$ and (4.15) occurs only for $i = 2$ and it reduces to

$$-d\phi + \omega_{12} + \sin\phi\cos\frac{u}{2}dx_1 + \cos\phi\sin\frac{u}{2}dx_2 = 0.$$

Since $d\phi = \phi_{x_1}dx_1 + \phi_{x_2}dx_2$, it follows from the expression of ω_{12} that ϕ must satisfy

$$\phi_{x_1} = \frac{u_{x_2}}{2} + \sin\phi\cos\frac{u}{2} \qquad \text{and} \qquad \phi_{x_2} = \frac{u_{x_1}}{2} + \cos\phi\sin\frac{u}{2}. \tag{4.20}$$

This is an integrable system for ϕ since u satisfies the sine-Gordon equation. For any solution ϕ of (4.20), we get from (4.18) the closed form

$$\theta = \cos\phi\cos\frac{u}{2}dx_1 - \sin\phi\sin\frac{u}{2}dx_2.$$

By considering $x_1 = t$ to be the time variable, we have the conservation law $\theta = \cos\phi\cos\frac{u}{2}dt - \sin\phi\sin\frac{u}{2}dx_2$. Hence, $\int -\sin\phi\sin\frac{u}{2}dx_2$ is a conserved quantity whenever the function $\cos\phi\cos(u/2)$ is either periodic in x_2 or it decays appropriately when x_2 tends to infinity.

Remark. We observe that in Theorem 2.6, whenever the metric and the connection forms associated to the solutions of the IGSGE are analytic in a parameter, then (4.19) may provide infinitely many conservation laws in time. This occurs very often in the 2-dimensional case as one can see in the literature of differential equations that describe pseudo-spherical surfaces (see [**5**]-[**13**]).

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Universidade de Brasília - UnB, Departamento de Matemática, 70910-900, Brasília–DF, Brazil
Email address: K.Tenenblat@mat.unb.br

Institut CNRS Pauli, Vienna-Austria; Laboratoire Painlevé, Lille, France
Email address: Barbara.Tumpach@math.cnrs.fr