An Example of Banach and Hilbert manifold: the universal Teichmüller space

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1. Motivations

H^s-Diffeomorphisms groups of the circle. For $s > 3/2$, the group $\text{Diff}^s(S^1)$ of Sobolev class H^s diffeomorphisms of the circle is a $\mathcal{C}^\infty$-manifold modeled on the space of H^s-section of the tangent bundle TS^1 ([1]), or equivalently on the space of real H^s-function on S^1. It is a topological group in the sense that the multiplication $(f, g) \mapsto f \circ g$ is well-defined and continuous, the inverse $f \mapsto f^{-1}$ is continuous, the left translation L_γ by $\gamma \in \text{Diff}^s(S^1)$ applying f to $\eta \circ f$ is continuous, and the right translation R_γ by $\gamma \in \text{Diff}^s(S^1)$ applying f to $f \circ \eta$ is smooth. These results are consequences of the Sobolev Lemma which states that for a compact manifold of dimension n, the space of H^s-sections of a vector bundle E over M is contained, for $s > k + n/2$, in the space of $\mathcal{C}^k$-sections, and that the injection $H^s(E) \hookrightarrow \mathcal{C}^k(E)$ is continuous. In particular, for $s > 3/2$, $\text{Diff}^s(S^1)$ is the intersection of the space of $\mathcal{C}^1$-diffeomorphisms of the circle with the space $H^s(S^1, S^1)$ of H^s maps from S^1 into itself. Hence $\text{Diff}^s(S^1)$ is an open set of $H^s(S^1, S^1)$.

For the same reasons, the subgroup of $\text{Diff}^s(S^1)$ preserving three points in S^1, say $-1, -i$ and 1, is, for $s > 3/2$, a $\mathcal{C}^\infty$ manifold and a topological group modeled on the space of H^s-vector fields which vanish on $-1, -i$ and 1.

One may ask what happens for the critical value $s = 3/2$ and look for a group with some regularity and a manifold structure such that the tangent space at the identity is isomorphic to the space of $H^{\frac{3}{2}}$-vector fields vanishing at $-1, -i$ and 1 (or equivalently on any codimension 3 subspace of $H^{\frac{3}{2}}$). The universal Teichmüller space $T_0(1)$ defined below will verify these conditions.

$\text{Diff}^+(S^1)$ as a group of symplectomorphisms. Consider the Hilbert space $\mathcal{V} = H^{\frac{1}{2}}(S^1, \mathbb{R})/\mathbb{R}$ of real valued $H^{\frac{1}{2}}$ functions with mean-value zero. Each element $u \in \mathcal{V}$ can be written as

$$u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \quad \text{with} \quad u_0 = 0,\ u_{-n} = \overline{u_n} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |n||u_n|^2 < \infty.$$

Endow $\mathcal{V}$ with the symplectic form

$$\Omega(u,v) = \frac{1}{2\pi}\int_{S^1} u(x)\partial_x v(x)dx = -i\sum_{n\in\mathbb{Z}} n u_n \overline{v_n},$$

The group of orientation preserving $\mathcal{C}^\infty$-diffeomorphisms of the circle acts on $\mathcal{V}$ by

$$\varphi\cdot f = f\circ\varphi - \frac{1}{2\pi}\int_{S^1} f\circ\varphi,$$

preserving the symplectic form Ω. Note that the previous action is well-defined for any orientation preserving homeomorphism of S^1. Therefore one may ask what is the biggest subgroup of the orientation preserving homeomorphisms of the circle which preserves $\mathcal{V}$ and Ω. The answer is the group of quasisymmetric homeomorphisms of the circle defined below (Theorem 3.1 and Proposition 4.1 in [3]).

Teichmüller spaces of compact Riemann surfaces. Consider a compact Riemann surface Σ. The Teichmüller space $\mathcal{T}(\Sigma)$ of Σ is defined as the space of complex structures on Σ modulo the action by pull-back of the group of diffeomorphisms which are homotopic to the identity. It can be endowed with a Riemannian metric, called the Weil-Petersson metric, which is not complete. A point beyond which a geodesic can not be continued corresponds to the collapsing of a handle of the Riemann surface ([6]), hence yields to a Riemann surface with lower genus. One can ask for a Riemannian manifold in which all the Teichmüller spaces of compact Riemann surfaces with arbitrary genus inject isometrically. The answer will be the universal Teichmüller space endowed with a Hilbert manifold structure and its Weil-Petersson metric ([5]).

2. The universal Teichmüller space

Quasiconformal and quasisymmetric mappings. Let us give some definitions and basic facts on quasiconformal and quasisymmetric mappings.

Definition 1. An orientation preserving homeomorphism f of an open subset A in $\mathbb{C}$ is called quasiconformal if the following conditions are satisfied.

- f admits distributional derivatives $\partial_z f$, $\partial_{\bar{z}} f \in L^1_{loc}(A, \mathbb{C})$;
- there exists $0 \leq k < 1$ such that $|\partial_{\bar{z}} f(z)| \leq k|\partial_z f(z)|$ for every $z \in A$.

Such an homeomorphism is said to be K-quasiconformal, where $K = \frac{1+k}{1-k}$.

Example 1. For example, $f(z) = \alpha z + \beta\bar{z}$ with $|\beta| < |\alpha|$ is $\frac{|\alpha|+|\beta|}{\alpha-|\beta|}$-quasiconformal.

Denote by $L^\infty(A,\mathbb{C})$ the complex Banach space of bounded complex valued functions on an open subset $A \subset \mathbb{C}$.

Theorem 2 ([2]). *An orientation preserving homeomorphism f defined on an open set $A \subset \mathbb{C}$ is quasiconformal if and only if it admits distributional derivatives $\partial_z f$, $\partial_{\bar z} f \in L^1_{loc}(A,\mathbb{C})$ which satisfy*

$$\partial_{\bar z} f(z) = \mu(z)\partial_z f(z), \quad z \in A$$

for some $\mu \in L^\infty(A,\mathbb{C})$ with $\|\mu\|_\infty < 1$.

The function μ appearing in the previous theorem is called the Beltrami coefficient or the complex dilatation of f. Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$.

Theorem 3 (Ahlfors-Bers). *Given $\mu \in L^\infty(\mathbb{D},\mathbb{C})$ with $\|\mu\|_\infty < 1$, there exists a unique quasiconformal mapping $\omega_\mu : \mathbb{D} \to \mathbb{D}$ with Beltrami coefficient μ, extending continuously to $\overline{\mathbb{D}}$, and fixing $1, -1, i$.*

Definition 2. An orientation preserving homeomorphism η of the circle S^1 is called quasisymmetric if there is a constant $M > 0$ such that for every $x \in \mathbb{R}$ and every $|t| \leq \frac{\pi}{2}$

$$\frac{1}{M} \leq \frac{\tilde\eta(x+t) - \tilde\eta(x)}{\tilde\eta(x) - \tilde\eta(x-t)} \leq M,$$

where $\tilde\eta$ is the increasing homeomorphism on $\mathbb{R}$ uniquely determined by $0 \leq \tilde\eta(0) < 1$, $\tilde\eta(x+1) = \tilde\eta(x) + 1$, and the condition that it projects onto η.

Theorem 4 (Beurling-Ahlfors extension Theorem). *Let η be an orientation preserving homeomorphism of S^1. Then η is quasisymmetric if and only if it extends to a quasiconformal homeomorphism of the open unit disc $\mathbb{D}$ into itself.*

$T(1)$ as a Banach manifold. One way to construct the universal Teichmüller space is the following. Denote by $L^\infty(\mathbb{D})_1$ the unit ball in $L^\infty(\mathbb{D},\mathbb{C})$. By Ahlfors-Bers theorem, for any $\mu \in L^\infty(\mathbb{D})_1$, one can consider the unique quasiconformal mapping $w_\mu : \mathbb{D} \to \mathbb{D}$ which fixes $-1, -i$ and 1 and satisfies the Beltrami equation on $\mathbb{D}$

$$\frac{\partial}{\partial \bar z}\omega_\mu = \mu \frac{\partial}{\partial z}\omega_\mu.$$

Therefore one can define the following equivalence relation on $L^\infty(\mathbb{D})_1$. For $\mu, \nu \in L^\infty(\mathbb{D})_1$, set $\mu \sim \nu$ if $w_\mu|S^1 = w_\nu|S^1$. The universal Teichmüller space is defined by the quotient space

$$T(1) = L^\infty(\mathbb{D})_1/\sim .$$

Theorem 5 ([2]). *The space $T(1)$ has a unique structure of complex Banach manifold such that the projection map $\Phi : L^\infty(\mathbb{D})_1 \to T(1)$ is a holomorphic submersion.*

The differential of Φ at the origin $D_0\Phi \; : L^\infty(\mathbb{D}, \mathbb{C}) \rightarrow T_{[0]}T(1)$ is a complex linear surjection and induces a splitting of $L^\infty(\mathbb{D}, \mathbb{C})$ into ([5]) :

$$L^\infty(\mathbb{D}, \mathbb{C}) = \operatorname{Ker} D_0\Phi \oplus \Omega_\infty(\mathbb{D}),$$

where $\Omega^\infty(\mathbb{D})$ is the Banach space defined by

$$\Omega_\infty(\mathbb{D}) := \left\{ \mu \in L^\infty(\mathbb{D}, \mathbb{C}) \mid \mu(z) = (1 - |z|^2)^2\overline{\phi(z)}, \quad \phi \text{ holomorphic on } \mathbb{D} \right\}.$$

$T(1)$ as a group. By the Beurling-Ahlfors extension theorem, a quasiconformal mapping on $\mathbb{D}$ extends to a quasisymmetric homeomorphism on the unit circle. Therefore the following map is a well-defined bijection

$$\begin{array}{ccc} T(1) & \rightarrow & \mathrm{QS}(S^1)/PSU(1,1) \\ [\mu] & \mapsto & \left[w_\mu | S^1\right]. \end{array}$$

The coset $\mathrm{QS}(S^1)/PSU(1,1)$ inherits from its identification with $T(1)$ a Banach manifold structure. Moreover the coset $\mathrm{QS}(S^1)/PSU(1,1)$ can be identified with the subgroup of quasisymmetric homeomorphisms fixing $-1, i$ and 1. This identification allows to endow the universal Teichmüller space with a group structure. Relative to this differential structure, the right translations in $T(1)$ are biholomorphic mappings, whereas the left translations are not even continuous in general. Consequently $T(1)$ is not a topological group.
The WP-metric and the Hilbert manifold structure on $T(1)$. The Banach manifold $T(1)$ carries a Weil-Petersson metric, which is defined only on a distribution of the tangent bundle ([4]). In order to resolve this problem the idea in [5] is to change the differentiable structure of $T(1)$.

Theorem 6 ([5])**.** *The universal Teichmüller space $T(1)$ admits a structure of Hilbert manifold on which the Weil-Petersson metric is a right-invariant strong hermitian metric.*

For this Hilbert manifold structure, the tangent space at [0] in $T(1)$ is isomorphic to

$$\Omega_2(\mathbb{D}) := \left\{ \mu(z) = (1 - |z|^2)^2\overline{\phi(z)}, \quad \phi \text{ holomorphic on } \mathbb{D}, \quad \|\mu\|_2 < \infty \right\},$$

where $\|\mu\|_2^2 = \int\int_{\mathbb{D}} |\mu|^2 \rho(z) d^2z$ is the L^2-norm of μ with respect to the hyperbolic metric of the Poincaré disc $\rho(z)d^2z = 4(1 - |z|^2)^{-2}d^2z$. The Weil-Petersson metric on $T(1)$ is given at the tangent space at $[0] \in T(1)$ by

$$\langle \mu, \nu \rangle_{WP} := \iint_{\mathbb{D}} \mu \overline{\nu} \, \rho(z) d^2z$$

With respect to this Hilbert manifold structure, $T(1)$ admits uncountably many connected components. For this Hilbert manifold structure, the identity component $T_0(1)$ of $T(1)$ is a topological group. Moreover the pull-back of the Weil-Petersson metric on the quotient space $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$ is given at [Id] by

$$h_{WP}([\mathrm{Id}])([u],[v]) = 2\pi \sum_{n=2}^{\infty} n(n^2 - 1) u_n \overline{v_n}.$$

Hence the identity component $T_0(1)$ of $T(1)$ can be seen as the completion of $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$ for the $H^{3/2}$-norm. This metric make $T(1)$ into a strong Kähler-Einstein Hilbert manifold, with respect to the complex structure given at [Id] by the Hilbert transform (see below where the definition of the Hilbert transform is recalled). The tangent space at [Id] consists of Sobolev class $H^{3/2}$ vector fields modulo $\mathfrak{psu}(1,1)$. The associated Riemannian metric is given by

$$\mathrm{g}_{WP}([\mathrm{Id}])([u],[v]) = \pi \sum_{n\neq -1,0,1} |n|(n^2-1)u_n\overline{v_n},$$

and the imaginary part of the Hermitian metric is the two-form

$$\omega_{WP}([\mathrm{Id}])([u],[v]) = -i\pi \sum_{n\neq -1,0,1} n(n^2-1)u_n\overline{v_n}.$$

Note that ω_{WP} coincides with the Kirillov-Kostant-Souriau symplectic form obtained on $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$ when considered as a coadjoint orbit of the Bott-Virasoro group.

3. The restricted Siegel disc

The Siegel disc. Let $\mathcal{V} = H^{\frac{1}{2}}(S^1,\mathbb{R})/\mathbb{R}$ be the Hilbert space of real valued $H^{\frac{1}{2}}$ functions with mean-value zero. The Hilbert inner product on $\mathcal{V}$ is given by

$$\langle u, v\rangle_{\mathcal{V}} = \sum_{n\in\mathbb{Z}} |n| u_n \overline{v_n}.$$

Endow the real Hilbert space $\mathcal{V}$ with the following complex structure (called the Hilbert transform)

$$\mathrm{J}\left(\sum_{n\neq 0} u_n e^{inx}\right) = i\sum_{n\neq 0}\mathrm{sgn}(n)u_n e^{inx}.$$

Now $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and J are compatible in the sense that J is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{V}}$. The associated symplectic form is defined by

$$\Omega(u,v) = \langle u, J(v)\rangle_{\mathcal{V}} = \frac{1}{2\pi}\int_{S^1} u(x)\partial_x v(x)dx = -i\sum_{n\in\mathbb{Z}} n u_n\overline{v_n}.$$

Let us consider the **complexified Hilbert space** $\mathcal{H} := H^{1/2}(S^1,\mathbb{C})/\mathbb{C}$ and the complex linear extensions of J and Ω still denoted by the same letters. Each element $u\in\mathcal{H}$ can be written as

$$u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \quad \text{with} \quad u_0 = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |n||u_n|^2 < \infty.$$

The eigenspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of the operator J are the following subspaces

$$\mathcal{H}_+ = \left\{u\in\mathcal{H}\,\middle|\, u(x) = \sum_{n=1}^{\infty} u_n e^{inx}\right\} \quad \text{and} \quad \mathcal{H}_- = \left\{u\in\mathcal{H}\,\middle|\, u(x) = \sum_{n=-\infty}^{-1} u_n e^{inx}\right\},$$

and one has the Hilbert decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into the sum of closed orthogonal subspaces. **The Siegel disc** associated with $\mathcal{H}$ is defined by

$$\mathfrak{D}(\mathcal{H}) := \{Z \in L(\mathcal{H}_-, \mathcal{H}_+) \mid \Omega(Zu, v) = \Omega(Zv, u),\ \forall\, u, v \in \mathcal{H}_- \quad \text{and} \quad I - Z\bar{Z} > 0\},$$

where, for $A \in L(\mathcal{H}_+, \mathcal{H}_+)$, the notation $A > 0$ means $\langle A(u), u\rangle_{\mathcal{H}} > 0$, for all $u \in \mathcal{H}_+, u \neq 0$ and where for $B \in L(\mathcal{H}_-, \mathcal{H}_+)$, define

$$\overline{B}(u) := \overline{B(\bar{u})}, \qquad B^T := (\bar{B})^*.$$

It follows easily that $\mathfrak{D}(\mathcal{H})$ can be written as

$$\mathfrak{D}(\mathcal{H}) := \{Z \in L(\mathcal{H}_-, \mathcal{H}_+) \mid Z^T = Z,\ \forall\, u, v \in \mathcal{H}_- \quad \text{and} \quad I - Z\bar{Z} > 0\}.$$

The restricted Siegel disc associated with $\mathcal{H}$ is by definition

$$\mathfrak{D}_{\text{res}}(\mathcal{H}) := \{Z \in \mathfrak{D}(\mathcal{H}) \mid Z \in L^2(\mathcal{H}_-, \mathcal{H}_+)\},$$

where $L^2(\mathcal{H}_-, \mathcal{H}_+)$ denotes the space of Hilbert-Schmidt operators from $\mathcal{H}_-$ to $\mathcal{H}_+$.

The restricted Siegel disc as an homogeneous space. Consider the symplectic group $\text{Sp}(\mathcal{V}, \Omega)$ of bounded linear maps on $\mathcal{V}$ which preserve the symplectic form Ω

$$\text{Sp}(\mathcal{V}, \Omega) = \{a \in \text{GL}(\mathcal{V}) \mid \Omega(au, av) = \Omega(u, v),\ \text{ for all } u, v \in \mathcal{V}\}.$$

The restricted symplectic group $\text{Sp}_{\text{res}}(\mathcal{V}, \Omega)$ is by definition the intersection of the symplectic group with the restricted general linear group defined by

$$\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}_+) = \left\{g \in \text{GL}(\mathcal{H}) \mid [d, g] \in L^2(\mathcal{H})\right\},$$

where $d := i(p_+ - p_-)$ and $p_\pm$ is the orthogonal projection onto $\mathcal{H}_\pm$. Using the block decomposition with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, one gets

$$\begin{aligned}&\text{Sp}_{\text{res}}(\mathcal{V}, \Omega)\\ &:= \left\{ \begin{pmatrix} g & h \\ \bar{h} & \bar{g}\end{pmatrix} \in \text{GL}(\mathcal{H}) \,\middle|\, h \in L^2(\mathcal{H}_-, \mathcal{H}_+), gg^* - hh^* = I, gh^T = hg^T \right\}.\end{aligned}$$

Proposition 7. *The restricted symplectic group acts transitively on the restricted Siegel disc by*

$$\text{Sp}_{\text{res}}(\mathcal{V}, \Omega) \times \mathfrak{D}_{\text{res}}(\mathcal{H}) \longrightarrow \mathfrak{D}_{\text{res}}(\mathcal{H}), \quad \left(\begin{pmatrix} g & h \\ \bar{h} & \bar{g}\end{pmatrix}, Z\right) \longmapsto (gZ + h)(\bar{h}Z + \bar{g})^{-1}.$$

The isotropy group of $0 \in \mathfrak{D}_{\text{res}}(\mathcal{H})$ *is the unitary group* $\text{U}(\mathcal{H}_+)$ *of* $\mathcal{H}_+$*, and the restricted Siegel disc is diffeomorphic as Hilbert manifold to the homogeneous space* $\text{Sp}_{\text{res}}(\mathcal{V}, \Omega)/U(\mathcal{H}_+)$*.*

On the space $\{A \in L^2(H_-, H_+) \mid A^T = A\}$ consider the following Hermitian inner product

$$\text{Tr}(V^*U) = \text{Tr}(\bar{V}U).$$

Since it is invariant under the isotropy group of $0 \in \mathfrak{D}_{\text{res}}(\mathcal{H})$, it extends to an $\text{Sp}_{\text{res}}(\mathcal{V}, \Omega)$-invariant Hermitian metric $h_{\mathfrak{D}}$.

Remark 8. In the construction above, replace $\mathcal{V}$ by $\mathbb{R}^2$ endowed with its natural symplectic structure. The corresponding Siegel disc is nothing but the open unit disc $\mathbb{D}$. The action of $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$ is the standard action of $\mathrm{SU}(1,1)$ on $\mathbb{D}$ given by

$$z \in \mathbb{D} \longmapsto \frac{az+b}{\bar{b}z+\bar{a}} \in \mathbb{D}, \quad |a|^2 - |b|^2 = 1,$$

and the Hermitian metric obtained on $\mathbb{D}$ is given by the hyperbolic metric

$$h_{\mathfrak{D}}(z)(u,v) = \frac{1}{(1-|z|^2)^2} u\bar{v}.$$

Therefore, $\mathfrak{D}_{\mathrm{res}}(\mathcal{H})$ can be seen as an infinite-dimensional generalization of the Poincaré disc.

4. The period mapping

The following theorems answer the second question adressed in the first section.

Theorem 9 (Theorem 3.1 in [3]). *For ϕ a orientation preserving homeomorphism and any $f \in \mathcal{V}$, set by $V_\phi f = f \circ \varphi - \frac{1}{2\pi}\int_{S^1} f \circ \varphi$. Then V_ϕ maps $\mathcal{V}$ into itself iff ϕ is quasisymmetric.*

Theorem 10 (Proposition 4.1 in [3]). *The group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle acts on the right by symplectomorphisms on $\mathcal{H} = H^{1/2}(S^1,\mathbb{C})/\mathbb{C}$ by*

$$V_\phi f = f \circ \varphi - \frac{1}{2\pi}\int_{S^1} f \circ \varphi,$$

$\varphi \in \mathrm{QS}(S^1)$, $f \in \mathcal{H}$.

Consequently this action defines a map $\Pi : \mathrm{QS}(S^1) \to \mathrm{Sp}(\mathcal{V},\Omega)$. Note that the operator $\Pi(\varphi)$ preserves the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ iff φ belongs to $\mathrm{PSU}(1,1)$. The resulting map (Theorem 7.1 in [3]) is an injective equivariant holomorphic immersion

$$\Pi \ : T(1) = \mathrm{QS}(S^1)/\,\mathrm{PSU}(1,1) \to \mathrm{Sp}(\mathcal{V},\Omega)/\,\mathrm{U}(H_+) \simeq \mathfrak{D}(\mathcal{H})$$

called the **period mapping** of $T(1)$. The Hilbert version of the period mapping is given by the following

Theorem 11 ([5]). *For $[\mu] \in T(1)$, $\Pi([\mu])$ belongs to the restricted Siegel disc if and only if $[\mu] \in T_0(1)$. Moreover the pull-back of the natural Kähler metric on $\mathfrak{D}_{\mathrm{res}}(\mathcal{H})$ coincides, up to a constant factor, with the Weil-Petersson metric on $T_0(1)$.*

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