On the classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits

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Abstract

In the finite-dimensional setting, every Hermitian-symmetric space of compact type is a coadjoint orbit of a finite-dimensional Lie group. It is natural to ask whether every infinite-dimensional Hermitian-symmetric space of compact type, which is a particular example of an Hilbert manifold, is transitively acted upon by a Hilbert Lie group of isometries. In this paper we give the classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits of simple connected L^*-groups of compact type using the notion of simple roots of non-compact type. The key step is, given an infinite-dimensional symmetric pair $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is a simple L^*-algebra of compact type and $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$, to construct an increasing sequence of finite-dimensional subalgebras $\mathfrak{g}_n$ of $\mathfrak{g}$ together with an increasing sequence of finite-dimensional subalgebras $\mathfrak{k}_n$ of $\mathfrak{k}$ such that $\mathfrak{g} = \overline{\cup \mathfrak{g}_n}$, $\mathfrak{k} = \overline{\cup \mathfrak{k}_n}$, and such that the pairs $(\mathfrak{g}_n, \mathfrak{k}_n)$ are symmetric. Comparing with the classification of Hermitian-symmetric spaces given by W. Kaup, it follows that any Hermitian-symmetric space of compact or non-compact type is an affine-coadjoint orbit of an Hilbert Lie group.

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1 Introduction

The topic of the present paper belongs to the theory of infinite-dimensional Hermitian-symmetric spaces, which are particular examples of symmetric spaces modelled on Banach spaces. The reader will find in [24] the fundamentals of the theory.

The classification of Hermitian-symmetric spaces of arbitrary dimension has been carried out by W. Kaup in [11] using the notion of Jordan triple systems developed in [10], and the equivalence between the category of simply connected, symmetric, complex Banach manifolds with base point and the category of Hermitian Jordan triple systems proved as the Main Theorem in [9]. A Hermitian-symmetric space $\mathcal{M}$ is defined to be a connected complex Banach manifold with a Hermitian structure such that each point in $\mathcal{M}$ is an isolated fixed point of an involutive holomorphic isometry of $\mathcal{M}$. By Theorem (4.2) in [11] and the discussion that follows, every Hermitian-symmetric space is the orthogonal product $\mathcal{M} = \mathcal{M}_+ \times \mathcal{M}_0 \times \mathcal{M}_-$ where $\mathcal{M}_0$ is the quotient of a Hilbert space by a discrete subgroup, and $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) is a simply-connected Hermitian-symmetric space of compact (resp. non-compact) type. By Theorem (3.9) and the discussion following Theorem (4.2) in [11], every Hermitian-symmetric space of compact (resp. non-compact) type is the orthogonal product of (possibly an infinite number of) irreducible Hermitian-symmetric spaces of compact (resp. non-compact) type. The category of irreducible Hermitian-symmetric spaces of compact type is equivalent to the category of irreducible Hermitian-symmetric spaces of non-compact type ([11]). It is therefore sufficient to classify either the irreducible Hermitian-symmetric spaces of compact type or the irreducible Hermitian-symmetric spaces of non-compact type.

In this paper, we are interested in the classification of irreducible infinite-dimensional Hermitian-symmetric affine coadjoint orbits of compact or non-compact type. In order to state the corresponding results, let us first introduce some notation. For any complex Hilbert space $\mathcal{F}$ endowed with a distinguished basis $\{f_j\}_{j\in J}$, $\mathcal{F}_{\mathbb{R}}$ will denote the real Hilbert space with basis $\{f_j\}_{j\in J}$ and $\mathcal{F}^{\mathbb{R}}$ the real

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Hilbert space with basis $\{f_j\}_{j\in J} \cup \{if_j\}_{j\in J}$. The Hilbert space of Hilbert-Schmidt operators on $\mathcal{F}$ will be denoted by $L^2(\mathcal{F})$, and the Banach space of trace class operators on $\mathcal{F}$ by $L^1(\mathcal{F})$. The group of invertible operators on $\mathcal{F}$ will be denoted by $\mathrm{GL}(\mathcal{F})$, and the group of unitary operators on $\mathcal{F}$ by $\mathrm{U}(\mathcal{F})$. In the sequel, $\mathcal{H}$ will denote a separable complex Hilbert space endowed with an orthonormal basis $\{\, e_n\}_{n\in\mathbb{Z}\setminus\{0\}}$. The Hermitian scalar product on $\mathcal{H}$ will be denoted by $\langle\cdot\,,\,\cdot\rangle_{\mathcal{H}}$ and will be $\mathbb{C}$-skew-linear with respect to the first variable, and $\mathbb{C}$-linear with respect to the second variable. For a bounded operator x on $\mathcal{H}$, denote by x^T the transpose of x defined by $\langle x^T e_i\,,\, e_j\rangle_{\mathcal{H}} = \langle x e_j\,,\, e_i\rangle_{\mathcal{H}}$, and by x^* the adjoint of x defined by $\langle x^* e_i\,,\, e_j\rangle_{\mathcal{H}} = \langle e_i\,,\, x e_j\rangle_{\mathcal{H}}$. The closed infinite-dimensional subspace of $\mathcal{H}$ generated by the e_n's for $n > 0$ will be called $\mathcal{H}_+$, and its orthogonal complement $\mathcal{H}_-$. For $0 < p < +\infty$, the p-dimensional subspace of $\mathcal{H}$ generated by the e_n's for $0 < n \leq p$ will be denoted $\mathcal{H}_p$. Let J_0 be the bounded operator on $\mathcal{H}$ defined by $J_0 e_i = -e_{-i}$ if $i < 0$ and $J_0 e_i = e_{-i}$ if $i > 0$. For $\mathcal{F} = \mathcal{H}, \mathcal{H}_\pm, \mathcal{H}_p$, or $\mathcal{H}_p^\perp$ define the following Hilbert Lie groups and the associated Lie algebras

$$\begin{array}{ll} \mathrm{GL}_2(\mathcal{F}) := \{g \in \mathrm{GL}(\mathcal{F}) \mid g - \mathrm{id} \in L^2(\mathcal{F})\}, & \mathfrak{gl}_2(\mathcal{F}) := L^2(\mathcal{F}), \\ \mathrm{U}_2(\mathcal{F}) := \{g \in \mathrm{U}(\mathcal{F}) \mid g - \mathrm{id} \in L^2(\mathcal{F})\}, & \mathfrak{u}_2(\mathcal{F}) := \{a \in L^2(\mathcal{F}) \mid a^* + a = 0\}, \\ \mathrm{O}_2(\mathcal{F}_{\mathbb{R}}) := \{g \in \mathrm{U}_2(\mathcal{F}) \mid g^T g = \mathrm{id}\}, & \mathfrak{o}_2(\mathcal{F}_{\mathbb{R}}) := \{a \in \mathfrak{u}_2(\mathcal{F}) \mid a^T + a = 0\} \end{array}$$

At last define

$$\mathrm{Sp}_2(\mathcal{H}) := \{g \in \mathrm{U}_2(\mathcal{H}) \mid g^T J_0 g = J_0\}, \quad \mathfrak{sp}_2(\mathcal{H}) := \{a \in \mathfrak{u}_2(\mathcal{H}) \mid a^T J_0 + J_0 a = 0\}.$$

On the Lie algebras $\mathfrak{g}$ listed above, the bracket is the commutator of operators and the Hermitian product $\langle\cdot\,,\,\cdot\rangle_{\mathcal{H}}$ is defined using the trace by

$$\langle A, B\rangle := \mathrm{Tr}\, A^* B.$$

These Lie algebras are L^*-algebras in the sense that the following property is satisfied :

$$\langle [x\,,\, y]\,,\, z\rangle = \langle y\,,\, [x^*,\, z]\rangle$$

for every x, y and z. In fact, $\mathfrak{u}_2(\mathcal{H})$, $\mathfrak{o}_2(\mathcal{H}_{\mathbb{R}})$ and $\mathfrak{sp}_2(\mathcal{H})$ are the only separable infinite-dimensional simple L^*-algebras of compact type modulo isomorphisms (see below for the corresponding definition and [1], [8], or [23] for the proof of this statement). An L^*-group is a Banach-Lie group whose Lie algebra has a structure of L^*-algebra (see [7]). The L^*-groups $\mathrm{GL}_2(\mathcal{H})$, $\mathrm{U}_2(\mathcal{H})$ and $\mathrm{Sp}_2(\mathcal{H})$ are connected, but $\mathrm{O}_2(\mathcal{H}_{\mathbb{R}})$ admits two connected components (see Proposition 12.4.2 on page 245 in [15]). The connected component of $\mathrm{O}_2(\mathcal{H}_{\mathbb{R}})$ containing the special orthogonal group

$$SO_1(\mathcal{H}_{\mathbb{R}}) := \{g \in \mathrm{O}_2(\mathcal{H}_{\mathbb{R}}) \mid g - \mathrm{id} \in L^1(\mathcal{H}), \det(g) = 1\},$$

where det denotes the Fredholm determinant (see [18]), will be denoted by $\mathrm{O}_2^+(\mathcal{H})$. The aim of this paper is to prove the following statement.

Theorem 1.1 *Every irreducible infinite-dimensional Hermitian-symmetric affine coadjoint orbit of a connected simple L^*-group of compact type is isomorphic to one of the following homogeneous space*

1. *the Grassmannian* $\mathrm{Gr}^{(p)} = \mathrm{U}_2(\mathcal{H})/\left(\mathrm{U}_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp)\right)$ *of p-dimensional subspaces of $\mathcal{H}$ with* $\dim(\mathcal{H}_p) = p < +\infty$
2. *the connected component of the restricted Grassmannian* $\mathrm{Gr}^0_{\mathrm{res}} = \mathrm{U}_2(\mathcal{H})/\left(\mathrm{U}_2(\mathcal{H}_+) \times \mathrm{U}_2(\mathcal{H}_-)\right)$ *of the polarized Hilbert space* $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ *with* $\dim\mathcal{H}_+ = \dim\mathcal{H}_- = +\infty$
3. *the Grassmannian* $\mathrm{Gr}^{(2)}_{\mathrm{or}} = O_2^+(\mathcal{H}_{\mathbb{R}})/\left(SO((\mathcal{H}_2)_{\mathbb{R}}) \times O_2^+((\mathcal{H}_2)^\perp_{\mathbb{R}})\right)$ *of oriented 2-planes in* $\mathcal{H}_{\mathbb{R}}$,
4. *the Grassmannian* $\mathcal{Z}(\mathcal{H}) = O_2^+(\mathcal{H}^{\mathbb{R}})/U_2(\mathcal{H})$ *of orientation-preserving orthogonal complex structures close to the distinguished complex structure on* $\mathcal{H}$,
5. *the Grassmannian* $\mathcal{L}(\mathcal{H}) = \mathrm{Sp}_2(\mathcal{H})/\mathrm{U}_2(\mathcal{H}_+)$ *of Lagrangian subspaces close to* $\mathcal{H}_+$.

Since there is a duality between affine coadjoint orbits of compact and non-compact type, Theorem 1.1 gives as a Corollary the classification of every irreducible infinite-dimensional Hermitian-symmetric affine coadjoint orbits of the connected L^*-groups of non-compact type with simple complexification. Each of these non-compact duals are symmetric Hilbert domains (see Corollary 3.17).

In the finite-dimensional case, every Hermitian-symmetric space of compact type is a coadjoint orbit of its connected group of isometries (see Proposition 8.89 in [4]). In the infinite-dimensional setting, the biggest group of isometries of a given Hermitian-symmetric space is not a Hilbert Lie group in general. For example the restricted unitary group $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ (see [15] for its definition) is a Banach Lie group acting by isometries on the restricted Grassmannian. It is a non trivial fact that the unitary Hilbert Lie group $\mathrm{U}_2(\mathcal{H})$, strictly contained in $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, acts transitively on each connected components of the restricted Grassmannian (see Proposition 5.2 in [3]). Theorem 1.1 above compared to the work of W. Kaup ([9], [10], [11]), leads to the following generalization :

Corollary 1.2 *Every Hermitian-symmetric space of compact or non-compact type is an homogeneous space of an Hilbert Lie group. More precisely, every Hermitian-symmetric space of compact or non-compact type is an affine-coadjoint orbit of an L^*-group.*

2 Root Theory of complex L^*-algebra

The root theory of complex L^*-algebras has been developed by J. R. Schue in [16] and [17]. Let us first recall that an L^*-algebra $\mathfrak{g}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a Lie algebra over $\mathbb{K}$, which is also a Hilbert space over $\mathbb{K}$ such that for every element $x \in \mathfrak{g}$, there exists $x^* \in \mathfrak{g}$ with the following property

$$\langle [x, y], z\rangle = \langle y, [x^*, z]\rangle, \tag{1}$$

for every y, z in $\mathfrak{g}$. In the case when $\mathbb{K} = \mathbb{C}$, our convention for the Hermitian product $\langle \cdot\,, \cdot\rangle$ is that it is $\mathbb{C}$-skew-linear with respect to the first variable, and $\mathbb{C}$-linear with respect to the second variable. The first example of L^*-algebra is a semi-simple finite-dimensional complex Lie algebra $\mathfrak{g}_0$ endowed with an involution σ, which defines a compact real form of $\mathfrak{g}_0$. In this example, the involutions $*$ and σ are related by $x^* = -\sigma(x)$ and the Hermitian scalar product is given by $\langle x, y\rangle = B(x^*, y)$, where B denotes the Killing form of $\mathfrak{g}_0$. An L^*-algebra is called of compact type if $x^* = -x$ for every x in $\mathfrak{g}$. It is called of non-compact type otherwise. For a given L^*-algebra $\mathfrak{g}$ the subspace

$$\mathfrak{k} := \{x \in \mathfrak{g} \mid x^* = -x\}$$

is a real L^*-algebra of compact type. Thus a complex L^*-algebra can be thought as an Hilbert Lie algebra together with a distinguished compact real form.

For every subsets A and B of an L^*-algebra $\mathfrak{g}$, $[A, B]$ will denote the *closure* of the vector space spanned by $\{[a, b] \mid a \in A, b \in B\}$. With this notation, an L^*-algebra is called semi-simple if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and simple if $\mathfrak{g}$ is non-commutative and if every closed ideal of $\mathfrak{g}$ is trivial. Every L^*-algebra can be decomposed into an orthogonal sum of its center and a semi-simple closed ideal (see [19], 2.2.13.). A Cartan subalgebra of a complex semi-simple L^*-algebra $\mathfrak{g}^{\mathbb{C}}$ is defined as a maximal Abelian $*$-stable subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Note that the condition of being $*$-stable is added in comparison to the finite-dimensional setting, hence a Cartan subalgebra may not be maximal in the set of Abelian subalgebras. It is noteworthy that a Cartan subalgebra of an L^*-algebra is in fact maximal Abelian (see [17], 1.1). Remark that a finite-dimensional Cartan subalgebra of a complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (for the usual definition) is contained in a compact real form of $\mathfrak{g}^{\mathbb{C}}$, thus is also a Cartan subalgebra of the corresponding finite-dimensional L^*-algebra. The existence of Cartan subalgebras of L^*-algebra is guarantied by Zorn's Lemma. Every semi-simple L^*-algebra is an Hilbert sum of closed $*$-stable simple ideals (see Theorem 1 in [16] for the complex case and Theorem 1 in [2] for the real case).

In the sequel, $\mathfrak{g}^{\mathbb{C}}$ will denote a semi-simple complex L^*-algebra and $\mathfrak{h}^{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. A root of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ is defined, as in the finite dimensional case, as an element α in the dual of $\mathfrak{h}^{\mathbb{C}}$ such that the corresponding "eigenspace"

$$V_\alpha \ := \{v \in \mathfrak{g}^{\mathbb{C}} \mid \forall h \in \mathfrak{h}^{\mathbb{C}}, [h, v] = \alpha(h)v\}.$$

is non-empty. In the following the set of non-zero roots with respect to a given Cartan subalgebra will be denoted by $\mathcal{R}$. Let us remark that a root has operator norm less than 1 and that for a non-zero root α, the vector space V_α is one-dimensional (see [16]). The Jacobi identity implies that

$$[V_\alpha, V_\beta] \subset V_{\alpha+\beta}. \tag{2}$$

By relation (1), $V_\alpha^* = V_{-\alpha}$. The main achievement in [17] is to prove that a semi-simple complex L^*-algebra $\mathfrak{g}^\mathbb{C}$ admits a Cartan decomposition with respect to a given Cartan subalgebra $\mathfrak{h}^\mathbb{C}$ in the sense that $\mathfrak{g}^\mathbb{C}$ is the Hilbert sum

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \sum_{\alpha\in\mathcal{R}} V_\alpha. \tag{3}$$

Let us remark that in a separable L^*-algebra, the set of root is countable or finite.

By Zorn's Lemma, one can decompose the set $\mathcal{R}$ of non-zero roots into two disjoint subsets $\mathcal{R}_+$ and $\mathcal{R}_-$ such that $\alpha \in \mathcal{R}_+ \Leftrightarrow -\alpha \in \mathcal{R}_-$. Such a decomposition defines a strict partial ordering on $\mathcal{R}$ by

$$\alpha > \beta \Leftrightarrow \alpha - \beta > 0,$$

where we write $\alpha - \beta > 0$ for $\alpha - \beta \in \mathcal{R}^+$. The elements in $\mathcal{R}_+$ will be called *positive* roots. In the sequel, a decomposition $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ as before and the induced ordering on the set of non-zero roots will be identified.

For every positive root α, one can choose $e_\alpha \in V_\alpha$ such that $\|e_\alpha\| = 1$. Then $e_\alpha^* \in V_{-\alpha}$ and $\|e_\alpha^*\| = 1$. This choice made, we define $e_\alpha := e_{-\alpha}^*$ for $\alpha \in \mathcal{R}_-$, in order to have, for every $\alpha \in \mathcal{R}$, the following relation $e_\alpha^* = e_{-\alpha}$. By (3), the set $\{e_\alpha \mid \alpha \in \mathcal{R}\}$ is an Hilbert basis of $\left(\mathfrak{h}^\mathbb{C}\right)^\perp$, and by (2), $[e_\alpha, e_\alpha^*]$ belongs to $\mathfrak{h}^\mathbb{C}$. We define the following elements in the Cartan subalgebra $\mathfrak{h}^\mathbb{C}$:

$$h_\alpha := [e_\alpha, e_\alpha^*], \tag{4}$$

for $\alpha \in \mathcal{R}^+$. A positive root is called *simple* if it can not be written as the sum of two positive roots. The set of simple roots will be denoted by $\mathcal{S}$. A subset $\mathcal{N}$ of the set of non-zero roots $\mathcal{R}$ is called a *root system*, if it satisfies the following conditions :

1. $\alpha \in \mathcal{N} \Rightarrow -\alpha \in \mathcal{N}$,
2. $(\alpha, \beta \in \mathcal{N}$ and $\alpha + \beta \in \mathcal{R}) \Rightarrow \alpha + \beta \in \mathcal{N}$.

A subset $\mathcal{N} \subset \mathcal{R}$ is called *indecomposable* if it can not be written as the union of two orthogonal non-empty subsets. As in the classical theory, one has the following facts. The set $\mathcal{R}$ of non-zero roots of a simple L^*-algebra is indecomposable. If F is an indecomposable subset of the set of non-zero roots $\mathcal{R}$, then it generates a root system $\mathcal{N}_F$, which is again indecomposable. The simple L^*-algebra generated by $\{e_\alpha \mid \alpha \in \mathcal{N}_F\}$ will be denoted by $\mathfrak{g}(\mathcal{N}_F)$.

For the classification of Hermitian-symmetric affine coadjoint orbits given in next section, we will need the following results. They were proved by J. R. Schue in [16] in order to classify the complex simple infinite-dimensional L^*-algebras.

Proposition 2.1 ([16]) *For every finite subset F of the set of non-zero roots $\mathcal{R}$ of a simple L^*-algebra, there exists a* finite *indecomposable system of non-zero roots containing F.*

Theorem 2.2 ([16], 3.2) *Let $\mathfrak{g}^\mathbb{C}$ be a simple complex separable L^*-algebra and $\mathcal{R} = \{\alpha_i \mid i \in \mathbb{N}\setminus\{0\}\}$ the set of non-zero roots with respect to a given Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$. For every $n \in \mathbb{N}\setminus\{0\}$, set $F_n := \{\alpha_1, \dots, \alpha_n\}$. Then there exists a sequence $\{\mathcal{N}_n\}_{n\in\mathbb{N}\setminus\{0\}}$ of* finite *subsets of $\mathcal{R}$ such that*

1. *$F_n \subset \mathcal{N}_n \subset \mathcal{N}_{n+1}$;*
2. *$\mathcal{N}_n$ is a indecomposable root system;*
3. *$\mathcal{R} = \cup_{n\in\mathbb{N}\setminus\{0\}}\mathcal{N}_n$*
4. *the simple subalgebras $\mathfrak{g}(\mathcal{N}_n)$ generated by $\mathcal{N}_n$ form a strictly increasing sequence with*

$$\mathfrak{g}^\mathbb{C} = \overline{\cup_{n\in\mathbb{N}\setminus\{0\}}\mathfrak{g}(\mathcal{N}_n)};$$

5. *The simple complex finite-dimensional algebras $\mathfrak{g}(\mathcal{N}_n)$ are of the same Cartan type A, B, C or D.*

Proposition 2.3 ([16], 3.2) *Given a sequence $\{\mathcal{N}_n\}_{n\in\mathbb{N}\setminus\{0\}}$ as in the previous Theorem, there exists a* total *ordering on the vector space generated by the set of roots such that :*

1. *$\alpha > 0 \Rightarrow -\alpha < 0$;*
2. *$\alpha > 0, \beta > 0 \Rightarrow \alpha + \beta > 0$;*
3. *If $\alpha > 0$ and $\alpha \notin \mathcal{N}_n$ then $\alpha > \beta$ for all $\beta \in \mathcal{N}_n$;*

4. the induced ordering on $\mathcal{N}_n$ is a lexicographical ordering with respect to a basis of roots.

Proposition 2.4 ([16], 3.3) *Let $\mathcal{S}$ be the set of simple roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to the ordering defined in the previous Proposition. The following assertions hold :*

1. *$\mathcal{S} \cap \mathcal{N}_n$ is a complete system of simple roots of the finite-dimensional algebra $\mathfrak{g}(\mathcal{N}_n)$, i.e. every positive root α of $\mathcal{N}_n$ can be written as a linear combination of elements in $\mathcal{S} \cap \mathcal{N}_n$ with non-negative integral coefficients;*
2. *If α and β belong to $\mathcal{S}$, $\alpha - \beta$ is a root if and only if $\alpha = \beta$;*
3. *the elements in $\mathcal{S}$ are linearly independent on the reals and every positive root $\alpha \in \mathcal{R}_+$ is a linear combination of elements in $\mathcal{S}$ with non-negative integral coefficients which are all zero except for a finite number of them.*

3 Classification of irreducible Hermitian-symmetric affine coadjoint orbits

The classification of finite-dimensional Hermitian-symmetric coadjoint orbits using the notion of roots of non-compact type has been carry out by J. A. Wolf in [26]. In this section we use the same technique to classify Hermitian-symmetric affine coadjoint orbits of connected simple L^*-groups of compact type, and then deduce a classification result for Hermitian-symmetric affine coadjoint orbits of non-compact type. Affine coadjoint orbits have been introduced in particular by K.-H. Neeb in [12]. Given an L^*-group G with Lie algebra $\mathfrak{g}$, an affine coadjoint action of G is a continuous homomorphism Ad^*_θ from G into the affine group of transformations $\mathrm{Aff}(\mathfrak{g}') = \mathrm{GL}(\mathfrak{g}') \ltimes \mathfrak{g}'$ of the continuous dual $\mathfrak{g}'$ of $\mathfrak{g}$ such that $\mathrm{Ad}^*_\theta(g) = (\mathrm{Ad}^*(g), \theta(g))$, $g \in G$, where Ad^* is the usual linear coadjoint action. By derivation at the unit element $\mathbf{1} \in G$, it gives an affine coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}'$, i.e. an continuous homomorphism $\mathrm{ad}^*_\theta : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g}') = \mathfrak{gl}(\mathfrak{g}') \ltimes \mathfrak{g}'$ such that $\mathrm{ad}^*_\theta(x) = (\mathrm{ad}^*(x), d\theta_e(x))$, $x \in \mathfrak{g}$. If $d\theta_{\mathbf{1}}(x) = \omega(x, \cdot)$ for a continuous 2-cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$, then the orbits of the affine coadjoint action of G defined by θ are naturally symplectic (see Theorem 2.4 in [12]) with symplectic form :

$$\Omega_\beta \left(\mathrm{ad}^*_\theta(x)(\beta), \mathrm{ad}^*_\theta(y)(\beta)\right) = \beta\left([x, y]\right) - \omega(x, y),$$

where $x, y \in \mathfrak{g}$ and $\beta \in \mathfrak{g}'$.

Definition 3.1 An affine coadjoint orbit $\mathcal{O}$ of G is called *Hermitian-symmetric* if it has a G-invariant structure of Hermitian-symmetric space.

Remark 3.2 A Hermitian-symmetric affine coadjoint orbit $\mathcal{O}$ is in particular *(locally-)symmetric*, i.e. the Lie algebra $\mathfrak{g}$ of G splits into $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of the isotropy group K fixing a given point $o \in \mathcal{O}$ and $\mathfrak{m}$ is a K-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$ such that

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

(Ones also says that $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair). Consequently the Levi-Civita connection of an affine coadjoint Hermitian-symmetric orbit $\mathcal{O}$ is the homogeneous connection and every G-invariant tensor is parallel (see e.g. Proposition 1.9 in [21] and its proof). In particular, $\mathcal{O}$ is Kähler since the complex structure is G-invariant hence parallel.

Since we are interested in Hermitian-symmetric orbits, which by the previous remark are in particular symplectic, we will consider only affine coadjoint actions such that $d\theta_e(x) = \omega(x, \cdot)$ for some continuous 2-cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$. Since the bracket $\langle \cdot, \cdot \rangle$ on the L^*-algebra $\mathfrak{g}$ is non-degenerate, there exists an operator $\mathbb{D}$ on $\mathfrak{g}$ such that :

$$\omega(x, y) = \langle x^*, \mathbb{D}y \rangle,$$

for $x, y \in \mathfrak{g}$. Since ω is a cocycle, $\mathbb{D}$ is a derivation of the Lie algebra $\mathfrak{g}$. By Remark 2.5(d) in [12], it is sufficient to consider affine coadjoint orbits of $0 \in \mathfrak{g}'$.

In the sequel, $\mathfrak{g}$ will denote an infinite-dimensional separable simple L^*-algebra of compact type and G_0 a connected simple L^*-group with the Lie algebra $\mathfrak{g}$. According to [1], [8] or [23], $\mathfrak{g}$ can be realized as a subalgebra of the L^*-algebra $\mathfrak{gl}_2(\mathcal{H})$ consisting of Hilbert-Schmidt operators on a separable complex

Hilbert space $\mathcal{H}$. We may therefore assume $\mathfrak{g} \subseteq \mathfrak{gl}_2(\mathcal{H})$. By the duality $\mathfrak{g}' = \mathfrak{g}$ given by the trace, we can identify affine adjoint and affine coadjoint orbits of G_0.

Suppose that $\mathcal{O}$ is a Hermitian-symmetric affine adjoint orbit of G_0 for an action as above. Then it is in particular strongly Kähler and by Theorem 4.4 in [12], there exists $D \in B(\mathcal{H})$ satisfying $D^* = -D$ such that for every x in $\mathfrak{g}$, $\mathbb{D}x = [D, x]$, as well as a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ which is contained in $\ker \mathbb{D}$. To emphasize the relation between the orbit and the bounded operator D, we will often write $\mathcal{O} = \mathcal{O}_D$.

Now let G be the group of operators on $\mathcal{H}$ generated by the exponentials of operators in $\mathfrak{g}$, and $G^{\mathbb{C}}$ be the group of operators on $\mathcal{H}$ generated by the exponentials of operators in $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g} \subseteq \mathfrak{gl}_2(\mathcal{H})$. That is, G (resp. $G^{\mathbb{C}}$) is the connected $\mathbf{1}$-component of the classical Hilbert-Lie group whose Lie algebra is $\mathfrak{g}$ (resp. $\mathfrak{g}^{\mathbb{C}}$). Since the center of G reduces to $\{\mathbf{1}\}$, it follows that the corresponding adjoint action

$$\mathrm{Ad}_G \ : G \to \mathrm{Ad}(\mathfrak{g})$$

is an isomorphism of Lie groups, where $\mathrm{Ad}(\mathfrak{g})$ is the adjoint group of the Banach-Lie algebra $\mathfrak{g}$. Recall that the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ has the natural structure of a Banach-Lie group whose Lie algebra consists of all derivations of $\mathfrak{g}$, and $\mathrm{Ad}(\mathfrak{g})$ is the connected (integral) subgroup of $\mathrm{Aut}(\mathfrak{g})$ corresponding to the Lie subalgebra of inner derivations of $\mathfrak{g}$.

On the other hand we have the adjoint action

$$\mathrm{Ad}_{G_0} \ : G_0 \to \mathrm{Ad}(\mathfrak{g}).$$

This Lie group homomorphism is onto and its kernel is equal to the center Z_{G_0} of G_0. Since G_0 is a simple Lie group, it follows that Z_{G_0} is a discrete subgroup. Thus we get a covering homomorphism

$$\pi = (\mathrm{Ad}_G)^{-1} \circ \mathrm{Ad}_{G_0} : G_0 \to G \, (\hookrightarrow B(\mathcal{H}))$$

whose fiber over $\mathbf{1} \in G$ is precisely the center of G_0, and for every $D \in B(\mathcal{H})$ the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\ \pi\ } & G \\ \Big\downarrow{\scriptstyle \mathrm{Ad}_{G_0,\omega_D}} & & \Big\downarrow{\scriptstyle \mathrm{Ad}_{G,\omega_D}} \\ B(\mathcal{H}) & \xrightarrow{\ \mathrm{id}\ } & B(\mathcal{H}) \end{array}$$

is commutative. Here the vertical arrows stand for the corresponding affine coadjoint actions :

$$\mathrm{Ad}_{G_0,\omega_D}(g)X = \mathrm{Ad}_{G_0}(g)X + \pi(g)D\pi(g)^{-1} - D$$

for every $g \in G_0$ and $X \in \mathfrak{g}$, and

$$\mathrm{Ad}_{G,\omega_D}(g)X = \mathrm{Ad}_G(g)X + gDg^{-1} - D$$

for every $g \in G$ and $X \in \mathfrak{g}$.

Since $\pi \ : G_0 \to G$ is a covering map, it follows by the above commutative diagram that the affine coadjoint orbits of G_0 and the ones of G are the same. Thus it suffices to investigate the affine coadjoint orbits of G.

Abusing slightly the notation, we will sometimes denote $\mathbb{D}$ by $\mathrm{ad}(D)$. An alternative definition of $\mathcal{O}_D$ is

$$\mathcal{O}_D = \{gDg^{-1} - D \mid g \in G\},$$

and the affine adjoint action of G on $\mathfrak{g}$ is given by

$$g \cdot a = \mathrm{Ad}_G(g)(a) + gDg^{-1} - D$$

where $g \in G$ and $a \in \mathfrak{g}$. The subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ which fixes 0 is

$$\mathfrak{k} \ := \{x \in \mathfrak{g} \mid [D, x] = 0\}.$$

It is an L^*-subalgebra of $\mathfrak{g}$. Let K be the isotropy subgroup of G that fixes 0. Since G and $\mathcal{O}$ are connected, K is connected. We will denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, which is in particular K-invariant.

From the discussion above it follows that it suffices to consider Hermitian-symmetric orbits $\mathcal{O}_D$ of the connected $\mathbf{1}$-component G of the classical Hilbert-Lie group whose Lie algebra is the infinite-dimensional separable simple L^*-algebra of compact type $\mathfrak{g} \subseteq \mathfrak{gl}_2(\mathcal{H})$. Such an orbit is said to be *of compact type* and admits a dual *of non-compact type* in the following sense. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the decomposition of the Lie algebra of G as above, then $\mathfrak{g}^{\text{n.c.}} := \mathfrak{k} \oplus i\mathfrak{m}$ is a real L^*-subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Since $\mathfrak{g}^{\mathbb{C}}$ is supposed to be a subalgebra of $\mathfrak{gl}_2(\mathcal{H})$, one can define the connected L^*-group $G^{\text{n.c.}}$ generated by the exponentials of operators in $\mathfrak{g}^{\text{n.c.}}$. Then the dual of $\mathcal{O}_D$ is defined as the affine coadjoint orbit of $G^{\text{n.c.}}$ for the derivation D. Let $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$ denote the complexifications of $\mathfrak{k}$ and $\mathfrak{m}$ respectively. Note that $\mathfrak{g}^{\mathbb{C}}$ is the orthogonal sum of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$ with respect to the Hermitian product of the L^*-algebra $\mathfrak{g}^{\mathbb{C}}$.

Proposition 3.3 *Let $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ that is contained in* $\ker \operatorname{ad} D$*, and let*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \mathcal{R}} V_\alpha$$

be the associated Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$, where $\mathcal{R}$ denotes the set of non-zero roots with respect to $\mathfrak{h}^{\mathbb{C}}$. Suppose that $\mathcal{O}_D$ is Hermitian-symmetric. Then there exists two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{R}$ such that $\mathcal{A} \cup \mathcal{B} = \mathcal{R}$ and

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \textstyle\sum_{\alpha \in \mathcal{A}} V_\alpha, \quad \mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \mathcal{B}} V_\alpha.$$

□ **Proof of Proposition** 3.3 :
Since $\mathcal{O}_D$ is (locally-)symmetric, one has $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ with

$$[\mathfrak{k}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}] \subset \mathfrak{k}^{\mathbb{C}} \ ; \qquad [\mathfrak{k}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}] \subset \mathfrak{m}^{\mathbb{C}} \ ; \qquad [\mathfrak{m}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}] \subset \mathfrak{k}^{\mathbb{C}}.$$

Let v be a non-zero vector in V_α, and $v = v_0 + v_1$ his decomposition with respect to the direct sum $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$. For every $h \in \mathfrak{h}^{\mathbb{C}}$, one has

$$[h, v] = [h, v_0 + v_1] = \alpha(h)(v_0 + v_1) = \alpha(h)v_0 + \alpha(h)v_1 = [h, v_0] + [h, v_1].$$

Since $[\mathfrak{h}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}] \subset \mathfrak{k}^{\mathbb{C}}$ and $[\mathfrak{h}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}] \subset \mathfrak{m}^{\mathbb{C}}$, it follows that

$$[h, v_0] = \alpha(h)v_0 \quad \text{et} \quad [h, v_1] = \alpha(h)v_1.$$

But V_α is one-dimensional, hence either $v_0 = 0$, or $v_1 = 0$. Consequently V_α is contained either in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{m}^{\mathbb{C}}$. □

Proposition 3.4 *For every $\alpha \in \mathcal{R}$, there exists a constant $c_\alpha \in \mathbb{R}$ such that $[D, e_\alpha] = ic_\alpha \, e_\alpha$. Moreover $c_{-\alpha} = -c_\alpha$.*

□ **Proof of Proposition** 3.4 :
For every $\alpha \in \mathcal{R}$ and every $h \in \mathfrak{h}^{\mathbb{C}}$, one has

$$[h, [D, e_\alpha]] = [[h, D], e_\alpha] + [D, [h, e_\alpha]] = \alpha(h)\,[D, e_\alpha].$$

The space V_α being one-dimensional, it follows that $[D, e_\alpha]$ is proportional to e_α. Since D satisfies $D^* = -D$, one has, for every $\alpha \in \mathcal{R}$, the following relation

$$\langle [D, e_\alpha], e_\alpha \rangle = -\langle e_\alpha, [D, e_\alpha] \rangle = -\overline{\langle [D, e_\alpha], e_\alpha \rangle}.$$

Thus there exists a real constant c_α such that

$$[D, e_\alpha] = ic_\alpha \, e_\alpha$$

On the other hand,

$$[D, e_\alpha]^* = [e_\alpha^*, D^*] = -[e_\alpha^*, D] = [D, e_\alpha^*].$$

Whence

$$\langle [D, e_\alpha^*], e_\alpha^* \rangle = \langle e_\alpha, [D, e_\alpha^*]^* \rangle = \langle e_\alpha, [D, e_\alpha] \rangle = ic_\alpha.$$

Consequently $[D, e_\alpha^*] = -ic_\alpha e_\alpha^*$. □

Remark 3.5 Let us denote by $\mathfrak{m}_+$ (resp. $\mathfrak{m}_-$) the closed subspace of $\mathfrak{m}^{\mathbb{C}}$ generated by the e_α's, where α runs over the set of roots for which $c_\alpha > 0$ (resp. $c_\alpha < 0$). Let $\mathcal{B}_+$ (resp. $\mathcal{B}_-$) be the set of roots β in $\mathcal{B}$ such that $V_\beta \in \mathfrak{m}_+$ (resp. $V_\beta \in \mathfrak{m}_-$).

Definition 3.6 The affine coadjoint orbit $\mathcal{O}_{\mathbb{D}}$ is called *(isotropy-)irreducible* if $\mathfrak{m}$ is a non-zero irreducible K-module.

Proposition 3.7 *If the affine adjoint orbit $\mathcal{O}_D$ is irreducible, then $\mathfrak{m}_+$ and $\mathfrak{m}_-$ are irreducible Ad(K)-modules, and there exists a constant $c > 0$ such that $\operatorname{ad}(D)_{|\mathfrak{m}_+} = ic \operatorname{id}_{|\mathfrak{m}_+}$ and $\operatorname{ad}(D)_{|\mathfrak{m}_-} = -ic \operatorname{id}_{|\mathfrak{m}_-}$. In particular, the spectrum of $\operatorname{ad}(D)$ is $\{0, ic, -ic\}$, hence D admits exactly two distinct eigenvalues.*

□ **Proof of Proposition** 3.7 :
For every $k \in \mathfrak{k}$ and every $e_\alpha \in \mathfrak{m}_\pm$, one has

$$[D, [k, e_\alpha]] = [[D, k], e_\alpha] + [k, [D, e_\alpha]] = ic_\alpha[k, e_\alpha].$$

It follows that $[\mathfrak{k}, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm$ and that $\mathfrak{m}_\pm$ is stable under the adjoint action of K. Let us suppose that $\mathfrak{m}_+$ decomposes into a sum of two non-zero Ad(K)-modules $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then

$$\mathfrak{m}_- = \mathfrak{m}_1^* \oplus \mathfrak{m}_2^*,$$

and it follows that $\mathfrak{m}$ decomposes also into the sum of two non-zero Ad(K)-modules, namely $\mathfrak{g} \cap (\mathfrak{m}_1 \oplus \mathfrak{m}_1^*)$ and $\mathfrak{g} \cap (\mathfrak{m}_2 \oplus \mathfrak{m}_2^*)$. The orbit $\mathcal{O}_D$ being irreducible, $\mathfrak{m}$ is an irreducible Ad(K)-module and this leads to a contradiction. So the irreducibility of $\mathfrak{m}_\pm$ is proved. Let e_α be an element in $\mathfrak{m}_+$ and set $c = c_\alpha$:

$$[D, e_\alpha] = ic\, e_\alpha.$$

The kernel $\ker(D - ic)$ being an Ad(K)-module of $\mathfrak{m}_+$, one has $\operatorname{ad}(D)_{|\mathfrak{m}_+} = ic \operatorname{id}_{|\mathfrak{m}_+}$. The relation $c_{-\alpha} = -c_\alpha$ implies that $\operatorname{ad}(D)_{|\mathfrak{m}_-} = -ic \operatorname{id}_{|\mathfrak{m}_-}$. □

Definition 3.8 Given an ordering on the set of non-zero roots $\mathcal{R}$ of $\mathfrak{g}^{\mathbb{C}}$, a simple root ϕ is called of *non-compact type* (see [26]) if every root $\alpha \in \mathcal{R}$ is of the form

$$\alpha = \pm \sum_{\Psi \in \mathcal{S} \smallsetminus \{\phi\}} a_\Psi \Psi, \text{ where } a_\Psi \geq 0 \text{ for all } \Psi \in \mathcal{S} \smallsetminus \{\phi\},$$

or of the form

$$\alpha = \pm \left(\phi + \sum_{\Psi \in \mathcal{S} \smallsetminus \{\phi\}} a_\Psi \Psi \right), \text{ where } a_\Psi \geq 0 \text{ for all } \Psi \in \mathcal{S} \smallsetminus \{\phi\}.$$

Lemma 3.9 *Let $\mathcal{O}_D$ be a Hermitian-symmetric affine adjoint irreducible orbit of a simple L^*-algebra $\mathfrak{g}$, $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ contained in* $\ker \operatorname{ad} D$, *and*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \mathcal{A}} V_\alpha \oplus \sum_{\beta \in \mathcal{B}} V_\beta$$

be the associated Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$ with

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \mathcal{A}} V_\alpha, \text{ and } \mathfrak{m}^{\mathbb{C}} = \sum_{\beta \in \mathcal{B}} V_\beta.$$

For every ordering $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ on the set of roots, there exists a unique simple root ϕ belonging to $\mathcal{B}$.

△ **Proof of Lemma** 3.9:
Let $\{\phi_i, \Psi_j\}_{i \in I, j \in J}$ be the set of simple roots with ϕ_i in $\mathcal{B}$ and Ψ_j in $\mathcal{A}$. Let us suppose that I is empty. The relation $[\mathfrak{k}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}] \subset \mathfrak{k}^{\mathbb{C}}$ implies that every positive root belongs to $\mathcal{A}$ and consequently $\mathfrak{m} = \{0\}$, which contradicts the hypothesis that $\mathfrak{m}$ is a non-zero irreducible Ad(K)-module. Let ϕ be a simple root in $\mathcal{B}$. The closed vector space spanned by the adjoint action of $\mathfrak{k}$ on e_ϕ is a non-zero irreducible Ad(K)-submodule of $\mathfrak{m}^{\mathbb{C}}$. It follows that ϕ is necessarily unique. △

Lemma 3.10 *Under the hypothesis of Lemma 3.9, there exists an increasing sequence of finite indecomposable root systems* $\mathcal{N}_n$ *such that*

1. $\mathcal{R} = \cup_{n\in\mathbb{N}\smallsetminus\{0\}}\mathcal{N}_n$*;*
2. *all the finite-dimensional subalgebras* $\mathfrak{g}(\mathcal{N}_n)$ *generated by* $\mathcal{N}_n$ *belong to the same type A, B, C, or D and* $\mathfrak{g}^{\mathbb{C}}$ *is the closure of the union of the subalgebras* $\mathfrak{g}(\mathcal{N}_n)$*;*
3. ϕ *is a simple root of non-compact type for each subalgebra* $\mathfrak{g}(\mathcal{N}_n)$ *with respect to the ordering on the roots of* $\mathfrak{g}(\mathcal{N}_n)$ *induced by the ordering on* $\mathcal{R}$ *defined in Proposition 2.3.*

$\triangle$ **Proof of Lemma** 3.10:
Let $\{\alpha_1, \ldots, \alpha_n, \ldots\}$ be a numbering of the roots in $\mathcal{A}$. Set $F_n = \{\alpha_1, \ldots, \alpha_n\}$. Let us construct by induction an increasing sequence of finite indecomposable root systems $\mathcal{N}_n$ as follows. By Proposition 2.1, there exists a finite indecomposable root system $\mathcal{N}_1$ containing $\{\phi\} \cup F_1$. Suppose that $\mathcal{N}_{n-1}$ is constructed, then there exists a finite indecomposable root system $\mathcal{N}_n$ containing $F_n \cup \mathcal{N}_{n-1}$. Since every root in $\mathcal{B}$ is the sum of ϕ and roots in $\mathcal{A}$, $\mathcal{R} = \cup_{n\in\mathbb{N}\smallsetminus\{0\}}\mathcal{N}_n$. The sequence of finite-dimensional simple subalgebras $\mathfrak{g}(\mathcal{N}_n)$ generated by the root systems $\mathcal{N}_n$ is increasing and such that $\mathfrak{g}^{\mathbb{C}} = \overline{\cup_{n\in\mathbb{N}\smallsetminus\{0\}}\mathfrak{g}(\mathcal{N}_n)}$. Since there exists only 9 types of finite-dimensional simple algebras, at least one type occurs an infinite number of times. Since $\mathfrak{g}^{\mathbb{C}}$ is infinite-dimensional and since only the types A, B, C, or D corresponds to algebras of arbitrary dimension, at least one of the types A, B, C, or D occurs an infinite number of times. It follows that there exists a subsequence $\mathcal{N}_{n_k}$ of $\mathcal{N}_n$ such that all the subalgebras $\mathfrak{g}(\mathcal{N}_{n_k})$ are of the same type A, B, C, or D. Let $\mathcal{S}_{n_k}$ be the set of simple roots of $\mathfrak{g}(\mathcal{N}_{n_k})$ with respect to the ordering induced by the ordering on $\mathcal{R}$ defined in Proposition 2.3. By Proposition 2.4, $\mathcal{S}_{n_k} = \mathcal{S} \cap \mathfrak{g}(\mathcal{N}_{n_k})$, where $\mathcal{S}$ is the set of simple roots of $\mathfrak{g}^{\mathbb{C}}$. For every positive root γ in $\mathcal{N}_{n_k}$, there exists a finite sequence $\{\gamma_i, i = 1, \ldots, k\}$ of roots in $\mathcal{S}_{n_k}$ such that

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_k,$$

and such that the partial sums $\gamma_1 + \cdots + \gamma_j$, $1 \leq j \leq k$ are roots (see [5]). Hence the vector space V_γ is generated by

$$v = [e_{\gamma_k}, [e_{\gamma_{k-1}}, [e_{\gamma_{k-2}}, \ldots, [e_{\gamma_2}, e_{\gamma_1}] \ldots]]].$$

The orbit $\mathcal{O}_D$ being irreducible, $[D, e_\phi] = \epsilon_\phi\, ic\, e_\phi$ with $\epsilon_\phi = \pm 1$ if $V_\phi \subset \mathfrak{m}_\pm$ (resp. $\mathfrak{m}_-$). Whence

$$[D, v] = \text{card}\,(\{i, \gamma_i = \phi\})\, \epsilon_\phi\, ic\, v.$$

Since $\text{ad}(D)$ preserves $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{m}_+$ and $\mathfrak{m}_-$, it follows that for γ in $\mathcal{A} \cap \mathcal{R}_+$, $\text{card}\,(\{i, \gamma_i = \phi\}) = 0$ and for γ in $\mathcal{B} \cap \mathcal{R}_+$, $\text{card}\,(\{i, \gamma_i = \phi\}) = 1$. Consequently ϕ is of non-compact type. $\triangle$

Proposition 3.11 *Let* $\mathcal{O} = G/K$ *be a Hermitian-symmetric irreducible affine coadjoint orbit of an* L^**-group* G *of compact type, and* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ *the associated decomposition of the Lie algebra* $\mathfrak{g}$ *of* G*, where* $\mathfrak{k}$ *is the Lie algebra of the isotropy group* K*. Then there exists an increasing sequence of finite-dimensional subalgebras* $\mathfrak{g}_n$ *of* $\mathfrak{g}$*, of the same type A, B, C or D, and an increasing sequence of subalgebras* $\mathfrak{k}_n$ *of* $\mathfrak{k}$ *such that*

1. $\mathfrak{g} = \overline{\cup \mathfrak{g}_n}$
2. $\mathfrak{k} = \overline{\cup \mathfrak{k}_n}$
3. *for every* $n \in \mathbb{N} \smallsetminus \{0\}$*, the orthogonal complement* $\mathfrak{m}_n$ *of* $\mathfrak{k}_n$ *in* $\mathfrak{g}_n$ *satisfies*

$$[\mathfrak{k}_n, \mathfrak{m}_n] \subset \mathfrak{m}_n \qquad \textit{and} \qquad [\mathfrak{m}_n, \mathfrak{m}_n] \subset \mathfrak{k}_n,$$

hence $(\mathfrak{g}_n, \mathfrak{k}_n)$ *is a symmetric pair.*

$\square$ **Proof of Proposition** 3.11 :
This is a direct consequence of Lemma 3.10, with $\mathfrak{g}_n = \mathfrak{g} \cap \mathfrak{g}(\mathcal{N}_n)$ and $\mathfrak{k}_n = \mathfrak{k} \cap \mathfrak{g}(\mathcal{N}_n)$. $\square$

From the discussion above it follows that the classification of Hermitian-symmetric irreducible affine coadjoint orbits of L^*-groups of compact or non-compact type can be deduced from the knowledge of the simple roots of non-compact type of finite-dimensional simple complex algebras (see the proof of Theorem 1.1 below). A simple root of a simple finite-dimensional complex algebra is of non-compact

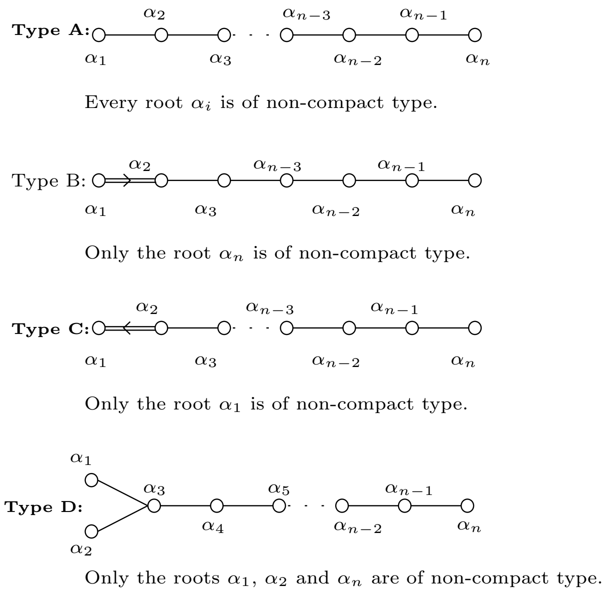

Table 1: *Simple roots of non-compact type in the simple finite-dimensional Lie algebras of type A, B, C and D.*

type if and only if it appears with the coefficient $+1$ in the expression of the greatest root. We recall the list of simple roots of non-compact type in the finite-dimensional Lie algebras of type A, B, C, or D in tabular 1 (see [6] or [26]).

■ **Proof of Theorem** 1.1 :
By Lemma 3.9, there exists a unique simple root ϕ in $\mathcal{B}$ regardless to the ordering chosen on the set of non-zero roots $\mathcal{R}$. By Lemma 3.10 part 3., ϕ is a simple root of non-compact type for each finite-dimensional subalgebras $\mathfrak{g}(\mathcal{N}_n)$ constructed in Lemma 3.10 part 2, when $\mathcal{R}$ is endowed with the particular ordering constructed in Proposition 2.3. For this ordering, simple roots of $\mathfrak{g}(\mathcal{N}_n)$ are simple roots of $\mathfrak{g}^{\mathbb{C}}$. It follows that the set of possible roots ϕ can be deduced from tabular 1. Such a root ϕ defines a unique symmetric pair of compact type $(\mathfrak{g}, \mathfrak{k})$ with $\mathfrak{g} = \{a \in \mathfrak{g}^{\mathbb{C}} \mid a + a^* = 0\}$, and $\mathfrak{k} = \{a \in \mathfrak{k}^{\mathbb{C}} \mid a + a^* = 0\}$ where $\mathfrak{k}^{\mathbb{C}}$ is the L^*-algebra whose Dynkin diagram is obtained by removing ϕ from the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ ($\mathfrak{k}^{\mathbb{C}}$ is the orthogonal complement of the vector space generated by the $e_{\phi+\alpha}$'s). One sees immediately that such a root ϕ defines also a unique symmetric pair of non-compact type, the dual of $(\mathfrak{g}, \mathfrak{k})$, namely $(\mathfrak{g}^{\text{n.c.}}, \mathfrak{k})$, where $\mathfrak{g}^{\text{n.c.}} := \mathfrak{k} \oplus i\mathfrak{m}$ and $\mathfrak{m}$ denotes the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$. ■

Example 3.12 The Grassmannian $\mathrm{Gr}^{(p)} = \mathrm{U}_2(\mathcal{H})/\big(\mathrm{U}_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^{\perp})\big)$ of p-dimensional subspaces of $\mathcal{H}$ with $\dim(\mathcal{H}_p) = p < +\infty$, is the affine adjoint orbit of $\mathrm{U}_2(\mathcal{H})$ for the derivations defined by the bounded operators $D_{k,l}^{(p)} = ik\,\mathrm{p}_{\mathcal{H}_p} - il\,\mathrm{p}_{\mathcal{H}_p^{\perp}}$, where $k, l \in \mathbb{R}$, $k \neq -l$, and $\mathrm{p}_{\mathcal{H}_p}$ (resp. $\mathrm{p}_{\mathcal{H}_p^{\perp}}$) is the orthogonal projection onto $\mathcal{H}_p$ (resp. $\mathcal{H}_p^{\perp}$). The homogeneous space $\mathrm{Gr}^{(p)}$ is therefore endowed with a one-parameter family of Kähler structures (encoded by $(k + l)$). The derivation $D_{k,l}^{(p)}$ is inner if and only if $l = 0$. For $p = 1$, $\mathrm{Gr}^{(p)}$ is the projective space of $\mathcal{H}$.

The dual symmetric space of $\mathrm{Gr}^{(p)}$ is the homogeneous space $\mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^{\perp})/\big(\mathrm{U}_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^{\perp})\big)$ where $\mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^{\perp})$ is the subgroup of $\mathrm{GL}_2(\mathcal{H})$ which preserves the indefinite Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathcal{H}$ defined by :

$$\langle\!\langle u, v \rangle\!\rangle = -\langle u_1, v_1 \rangle_{\mathcal{H}_p^{\perp}} + \langle u_2, v_2 \rangle_{\mathcal{H}_p},$$

where $u = u_1 + u_2$, $v = v_1 + v_2$ with $u_1, v_1 \in \mathcal{H}_p^{\perp}$ and $u_2, v_2 \in \mathcal{H}_p$. It is the affine adjoint orbit of $\mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^{\perp})$ for the derivations $D_{k,l}^{(p)}$. It can be identified with the symmetric Hilbert domain :

$$\mathcal{A}^{(p)} = \{Z \in L^2(\mathcal{H}_p, \mathcal{H}_p^{\perp}), -Z^*Z + \mathrm{id} > 0\},$$

where the notation $-Z^*Z + \mathrm{id} > 0$ means that the operator $-Z^*Z + \mathrm{id}$ is positive definite. In particular, for $p = 1$, $\mathcal{A}^{(1)}$ is the open unit ball in $\mathcal{H}_1^\perp$. Let us remark that $\mathcal{A}^{(p)}$ is star-shaped hence connected and simply-connected. To see that $\mathcal{A}^{(p)}$ is diffeomorphic to the homogeneous space $\mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^\perp)/\left(\mathrm{U}_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp)\right)$, note that

$$\mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^\perp) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathcal{H}) \mid A^*A - C^*C = \mathrm{id}_{\mathcal{H}_p^\perp}, D^*D - B^*B = \mathrm{id}_{\mathcal{H}_p}, A^*B = C^*D \right\},$$

where the block decomposition of g is relative to the Hilbert sum $\mathcal{H} = \mathcal{H}_p^\perp \oplus \mathcal{H}_p$. In particular, for $Z \in \mathcal{A}^{(p)}$, one has

$$-(AZ + B)^*(AZ + B) + (CZ + D)^*(CZ + D) = -Z^*Z + 1 > 0,$$

which implies that $(CZ + D)^*(CZ + D)$ is positive definite hence $(CZ + D) \in \mathrm{GL}(\mathcal{H}_p)$. It follows that one can define an action of $\mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^\perp)$ on $\mathcal{A}^{(p)}$ by :

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}, \tag{5}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_p^\perp \oplus \mathcal{H}_p$. This action is transitive since every $Z \in \mathcal{A}^{(p)}$ can be written as

$$Z = \exp \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \cdot 0,$$

where

$$B = Z \frac{\operatorname{argth}(Z^*Z)^{\frac{1}{2}}}{(Z^*Z)^{\frac{1}{2}}} \in L^2(\mathcal{H}_p, \mathcal{H}_p^\perp) \tag{6}$$

(this expression follows from Remark 6.5 in [13]). Another proof for the transitivity of the action (5) of $\mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^\perp)$ on $\mathcal{A}^{(p)}$ can be found in [14], Theorem III.9. Since the isotropy of $0 \in \mathcal{A}^{(p)}$ is $\mathrm{U}_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp)$, one has :

$$\mathcal{A}^{(p)} = \mathrm{U}_2(\mathcal{H}_p, \mathcal{H}_p^\perp)/\left(\mathrm{U}_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp)\right).$$

The Hermitian-symmetric space of non-compact type $\mathcal{A}^{(p)}$ is a particular example of Finsler-Cartan-Hadamard manifold (see the Definition on p 124, Proposition 3.16, Proposition 3.15, and Theorem 3.6(iii) in [13]). It follows either from the general theory (Theorem 3.14 or Theorem 1.10 in [13]) or from equation (6) that

$$\begin{array}{rccc} \exp \ : & \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \mid B \in L^2(\mathcal{H}_p, \mathcal{H}_p^\perp) \right\} & \longrightarrow & \mathcal{A}^{(p)} \\ & X & \longmapsto & \exp X \cdot 0 \end{array}$$

is a diffeomorphism.

Example 3.13 The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ has been studied in [15] and [27]. The connected component $\mathrm{Gr}^0_{\mathrm{res}}$ of $\mathrm{Gr}_{\mathrm{res}}$ containing $\mathcal{H}_+$ is the affine adjoint orbit of $\mathrm{U}_2(\mathcal{H})$ for the derivations defined by the bounded operators $D^{(\infty)}_{k,l} = ik\,\mathrm{p}_+ - il\,\mathrm{p}_-$, where $k, l \in \mathbb{R}$, $k \neq -l$, and $\mathrm{p}_\pm$ is the orthogonal projection onto $\mathcal{H}_\pm$. None of these derivations is inner.

As in the previous case, the dual Hermitian-symmetric space of the connected component $\mathrm{Gr}^0_{\mathrm{res}}$ of the restricted Grassmannian is the homogeneous space $\mathrm{U}_2(\mathcal{H}_+, \mathcal{H}_-)/(\mathrm{U}_2(\mathcal{H}_+) \times U_2(\mathcal{H}_-))$ where $\mathrm{U}_2(\mathcal{H}_+, \mathcal{H}_-)$ is the subgroup of $\mathrm{GL}_2(\mathcal{H})$ which preserves the indefinite Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathcal{H}$ defined by :

$$\langle\!\langle u, v \rangle\!\rangle = -\langle u_1, v_1 \rangle_{\mathcal{H}_-} + \langle u_2, v_2 \rangle_{\mathcal{H}_+},$$

where $u = u_1 + u_2$, $v = v_1 + v_2$ with $u_1, v_1 \in \mathcal{H}_-$ and $u_2, v_2 \in \mathcal{H}_+$. It is the affine adjoint orbit of $\mathrm{U}_2(\mathcal{H}_+, \mathcal{H}_-)$ for the derivations $D^{(\infty)}_{k,l}$. It can be identified with the symmetric Hilbert domain :

$$\mathcal{A}^{(\infty)} = \{Z \in L^2(\mathcal{H}_+, \mathcal{H}_-) \mid -Z^*Z + \mathrm{id} > 0\}.$$

Example 3.14 Denote by g the real part of the Hermitian scalar product on $\mathcal{H}$. The Grassmannian $\mathcal{Z}(\mathcal{H}) = O_2^+(\mathcal{H}^{\mathbb{R}})/U_2(\mathcal{H})$ is the space of complex structures I on $\mathcal{H}^{\mathbb{R}}$ such that

$$\mathrm{g}(IX, IY) = \mathrm{g}(X, Y),$$

defining the same orientation as the distinguished complex structure I_0 on $\mathcal{H}$ and being closed to it. For every $k \neq 0$, the space $\mathcal{Z}(\mathcal{H})$ can be identified with the $O_2^+(\mathcal{H}^{\mathbb{R}})$-affine adjoint orbit of 0 for the bounded operator $D_k^{(0)} = kI_0$. Denote by $\mathcal{H}^{\mathbb{C}}$ the $\mathbb{C}$-extension of $\mathcal{H}^{\mathbb{R}}$ and by $Z_\pm$ the eigenspace of the $\mathbb{C}$-linear extension of I_0 with eigenvalue $\pm i$. One has $\mathcal{H}^{\mathbb{C}} = Z_+ \oplus Z_-$ as orthogonal sum with respect to the Hermitian scalar product on $\mathcal{H}^{\mathbb{C}}$ which restricts to g on $\mathcal{H}^{\mathbb{R}}$. The homogeneous space $\mathcal{Z}(\mathcal{H})$ injects into the restricted Grassmannian of the polarized Hilbert space $\mathcal{H}^{\mathbb{C}} = Z_+ \oplus Z_-$ via the application which maps a complex structure I to the subspace of $\mathcal{H}^{\mathbb{C}}$ consisting of $(1,0)$-type vectors X with respect to I, i.e. satisfying $IX = iX$. This realizes $\mathcal{Z}(\mathcal{H})$ as the totally geodesic submanifold of $\mathrm{Gr}^0_{\mathrm{res}}$ consisting of maximal isotropic subspaces for the $\mathbb{C}$-linear extension $\mathrm{g}^{\mathbb{C}}$ of g. Starting with a basis $\{e_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ of $\mathcal{H}$, endow $\mathcal{H}^{\mathbb{R}}$ with the basis $\{e_n\}_{n\in\mathbb{Z}\setminus\{0\}} \cup \{I_0 e_n\}_{n\in\mathbb{Z}\setminus\{0\}}$. Then $Z_\pm$ is the $\mathbb{C}$-linear subspace of $\mathcal{H}^{\mathbb{C}}$ generated by $\{\frac{1}{\sqrt{2}}(e_n \mp iI_0 e_n\}_{n\in\mathbb{Z}\setminus\{0\}}$. With respect to these basis, the symmetric $\mathbb{C}$-bilinear form $\mathrm{g}^{\mathbb{C}}$ and the $\mathbb{C}$-linear extension of the operator $D_k^{(0)}$ have the following decompositions as endomorphisms of $\mathcal{H}^{\mathbb{C}} = Z_+ \oplus Z_-$:

$$\mathrm{g}^{\mathbb{C}} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix},$$
$$D_k^{(0)} = \begin{pmatrix} ik & 0 \\ 0 & -ik \end{pmatrix}.$$

It is easy to see that $O_2^+(\mathcal{H}^{\mathbb{R}})$ (as defined in the Introduction) is conjugate to the connected component of $O_2(\mathcal{H}^{\mathbb{C}}) \cap U_2(\mathcal{H}^{\mathbb{C}})$ where $O_2(\mathcal{H}^{\mathbb{C}})$ denotes the complex L^*-group preserving $\mathrm{g}^{\mathbb{C}}$.

The dual symmetric space of $\mathcal{Z}(\mathcal{H})$ is the homogeneous space $\left(O_2(\mathcal{H}^{\mathbb{C}}) \cap \mathrm{U}_2(Z_+, Z_-)\right)/U_2(Z_+)$ where $\mathrm{U}_2(Z_+, Z_-)$ is the subgroup of $\mathrm{GL}_2(\mathcal{H}^{\mathbb{C}})$ which preserves the indefinite Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathcal{H}^{\mathbb{C}}$ defined by :

$$\langle\!\langle u, v \rangle\!\rangle = -\langle u_1, v_1\rangle_{Z_-} + \langle u_2, v_2\rangle_{Z_+},$$

where $u = u_1 + u_2$, $v = v_1 + v_2$ with $u_1, v_1 \in Z_-$ and $u_2, v_2 \in Z_+$. It is the affine adjoint orbit of $O_2(\mathcal{H}^{\mathbb{C}}) \cap U_2(Z_+, Z_-)$ for the derivations $D_k^{(0)}$, $k \neq 0$. It can be identified with the symmetric Hilbert domain :

$$\mathcal{B}^{(\infty)} = \{Z \in L^2(Z_+, Z_-) \mid Z^T + Z = 0, -Z^*Z + \mathrm{id} > 0\}.$$

Example 3.15 The Grassmannian $\mathcal{L}(\mathcal{H}) = \mathrm{Sp}_2(\mathcal{H})/\mathrm{U}_2(\mathcal{H}_+)$ of Lagrangian subspaces close to $\mathcal{H}_+$ is the $\mathrm{Sp}_2(\mathcal{H})$-affine adjoint orbit of 0 for the derivations given by the bounded operators $D_{l,l}^{(\infty)} = il\,\mathrm{p}_+ - i\,l\mathrm{p}_-$, $l \neq 0$. It is a totally geodesic submanifold of the restricted Grassmannian $\mathrm{Gr}^0_{\mathrm{res}}$.

The dual symmetric space of $\mathcal{L}(\mathcal{H})$ is the homogeneous space $\mathrm{Sp}_2(\mathcal{H}, \mathbb{C}) \cap U_2(H_+, H_-)/\mathrm{U}_2(\mathcal{H}_+)$, where $\mathrm{Sp}_2(\mathcal{H}, \mathbb{C})$ is the complex L^*-group preserving the $\mathbb{C}$-bilinear antisymmetric form $\omega(X, Y) = X^T J_0 Y$. It is the affine adjoint orbit of $\mathrm{Sp}_2(\mathcal{H}, \mathbb{C}) \cap U_2(H_+, H_-)$ for the derivations $D_{l,l}^{(\infty)}$, $l \neq 0$. It can be identified with the symmetric Hilbert domain :

$$\mathcal{C}^{(\infty)} = \{Z \in L^2(Z_+, Z_-) \mid Z^T = Z, -Z^*Z + \mathrm{id} > 0\}.$$

Note that $\mathrm{Sp}_2(\mathcal{H}, \mathbb{C}) \cap U_2(H_+, H_-)$ is conjugate to

$$\mathrm{Sp}_2(\mathcal{H}_{\mathbb{R}}) := \{g \in \mathrm{GL}_2(\mathcal{H}_{\mathbb{R}}) \mid g^T J_0 g = J_0\},$$

hence $\mathcal{C}^{(\infty)} = \mathrm{Sp}_2(\mathcal{H}_{\mathbb{R}})/\mathrm{U}_2(\mathcal{H}_{\mathbb{R}}, J_0)$ where $\mathrm{U}_2(\mathcal{H}_{\mathbb{R}}, J_0)$ denotes the unitary group of the Hilbert space $\mathcal{H}_{\mathbb{R}}$ endowed with the complex structure J_0.

Example 3.16 Recall that $\mathcal{H}_{\mathbb{R}}$ is a real Hilbert space with basis $\{e_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ and that $(\mathcal{H}_2)_{\mathbb{R}}$ denotes the real subspace generated by e_1 and e_2. The space $\mathrm{Gr}^{(2)}_{\mathrm{or}} = O_2^+(\mathcal{H}_{\mathbb{R}})/\left(SO((\mathcal{H}_2)_{\mathbb{R}}) \times O_2^+((\mathcal{H}_2)^{\perp}_{\mathbb{R}})\right)$ is the Grassmannian of oriented 2-planes in $\mathcal{H}_{\mathbb{R}}$ and the $O_2^+(\mathcal{H}_{\mathbb{R}})$-adjoint orbit of kJ where $k \neq 0$ and J is the natural complex structure on $(\mathcal{H}_2)_{\mathbb{R}}$. Via the map which assigns to an oriented 2-plane of $\mathcal{H}_{\mathbb{R}}$ with orthonormal basis $\{u, v\}$ the complex line $\mathbb{C}(u + iv) \in \mathbb{P}(\mathcal{H})$, the Grassmannian $\mathrm{Gr}_{\mathrm{or}}^{(2)}$ can be identified as complex manifold with the quadric $\mathcal{C}$ in the complex projective space $\mathbb{P}(\mathcal{H})$ defined by

$$\mathcal{C} := \left\{ [z] = \left[\sum_{i \in \mathbb{Z} \smallsetminus \{0\}} z_i e_i \right] \in \mathbb{P}(\mathcal{H}) \mid \sum_{i \in \mathbb{Z} \smallsetminus \{0\}} z_i^2 = 0 \right\}.$$

The dual Hermitian-symmetric space of $\mathrm{Gr}_{\mathrm{or}}^{(2)}$ is the homogeneous space

$$O_2^+\left((\mathcal{H}_2)_{\mathbb{R}}, (\mathcal{H}_2)_{\mathbb{R}}^{\perp}\right) / \left(SO((\mathcal{H}_2)_{\mathbb{R}}) \times O_2^+((\mathcal{H}_2)_{\mathbb{R}}^{\perp})\right)$$

where $O_2^+\left((\mathcal{H}_2)_{\mathbb{R}}, (\mathcal{H}_2)_{\mathbb{R}}^{\perp}\right)$ is the subgroup of $\mathrm{GL}_2(\mathcal{H}_{\mathbb{R}})$ which preserves the indefinite symmetric form $(\!(\ , \ )\!)$ on $\mathcal{H}_{\mathbb{R}}$ defined by :

$$(\!(u, v)\!) = u_1 v_1 + u_2 v_2 - \sum_{i \in \mathbb{Z} \smallsetminus \{0,1,2\}} u_i v_i,$$

where $u = \sum_{i \in \mathbb{Z} \smallsetminus \{0\}} u_i e_i$ and $v = \sum_{i \in \mathbb{Z} \smallsetminus \{0\}} v_i e_i$. It is the $O_2^+\left((\mathcal{H}_2)_{\mathbb{R}}, (\mathcal{H}_2)_{\mathbb{R}}^{\perp}\right)$-adjoint orbit for the derivations given by the bounded operators kJ for $k \neq 0$. It can be identified with the symmetric Hilbert domain :

$$\mathcal{D}^2 = \{Z \in \mathcal{H}_2^{\perp} \mid 1 + |Z^T Z|^2 - 2Z^* Z > 0, -Z^* Z + 1 > 0\},$$

where $|Z^T Z|^2 = Z^* \bar{Z} Z^T Z$. (see [25] p 350–351).

Corollary 3.17 *Every infinite-dimensional irreducible Hermitian-symmetric affine (co-)adjoint orbit of a connected L^*-group of non-compact type with simple complexification is isomorphic to one of the following symmetric Hilbert domains :*

1. $\mathcal{A}^{(p)} = \{Z \in L^2(\mathcal{H}_p, \mathcal{H}_p^{\perp}) \mid -Z^* Z + \mathrm{id} > 0\}$;
2. $\mathcal{A}^{(\infty)} = \{Z \in L^2(\mathcal{H}_+, \mathcal{H}_-) \mid -Z^* Z + \mathrm{id} > 0\}$;
3. $\mathcal{B}^{(\infty)} = \{Z \in L^2(Z_+, Z_-) \mid -Z^* Z + \mathrm{id} > 0,\ Z^T + Z = 0\}$;
4. $\mathcal{C}^{(\infty)} = \{Z \in L^2(Z_+, Z_-) \mid -Z^* Z + \mathrm{id} > 0,\ Z^T = Z\}$;
5. $\mathcal{D}^2 = \{Z \in \mathcal{H}_2^{\perp} \mid -Z^* Z + 1 > 0,\ 1 + |Z^T Z|^2 - 2Z^* Z > 0\}$.

□ **Proof of Corollary** 3.17 :
Let $\mathcal{O}^{\mathrm{n.c.}}$ be an infinite-dimensional irreducible Hermitian-symmetric affine (co-)adjoint orbit of a non-compact L^*-group $G^{\mathrm{n.c.}}$ with simple complexification. Let $\mathfrak{g}^{\mathrm{n.c.}}$ be the Lie algebra of $G^{\mathrm{n.c.}}$. Since $(\mathfrak{g}^{\mathrm{n.c.}})^{\mathbb{C}}$ is simple, $\mathfrak{g}^{\mathrm{n.c.}}$ is itself simple. Moreover we can suppose w.l.o.g. ([16]) that $(\mathfrak{g}^{\mathrm{n.c.}})^{\mathbb{C}}$ is either $\mathfrak{gl}_2(\mathcal{H})$, $\mathfrak{o}_2(\mathcal{H}^{\mathbb{C}})$ or $\mathfrak{sp}_2(\mathcal{H}, \mathbb{C})$, where $\mathfrak{o}_2(\mathcal{H}^{\mathbb{C}})$ (resp. $\mathfrak{sp}_2(\mathcal{H}, \mathbb{C})$) is the Lie algebra of $O_2(\mathcal{H}^{\mathbb{C}})$ (resp. $\mathrm{Sp}_2(\mathcal{H}, \mathbb{C})$) introduced in Example 3.14 (resp. in Example 3.15).

Since $\mathcal{O}^{\mathrm{n.c.}}$ is in particular strongly symplectic, by Theorem 4.4 in [12], the derivation defining $\mathcal{O}^{\mathrm{n.c.}}$ can be written as $\mathbb{D}x = [D, x]$ where D is a skew-Hermitian operator with finite spectrum. It follows that $\mathfrak{k} := \mathrm{Ker}\,\mathbb{D}$ is $*$-invariant. Since $\mathfrak{g}^{\mathrm{n.c.}}$ is in particular semi-simple, one has $\langle x, y \rangle = \langle y^*, x^* \rangle$ for every x, y in $\mathfrak{g}^{\mathrm{n.c.}}$. Hence the orthogonal complement $\mathfrak{n}$ of $\mathfrak{k}$ in $\mathfrak{g}^{\mathrm{n.c.}}$ is also $*$-invariant. Denote by K the isotropy subgroup of $G^{\mathrm{n.c.}}$ fixing 0. Since $\mathfrak{n}$ is an irreducible K-module, the bilinear form b on $\mathfrak{n}$ defining the Riemannian metric of $\mathcal{O}^{\mathrm{n.c.}}$ is proportional to the trace, that is : $b(x, y) = \lambda \mathrm{Tr} xy$, for x, y in $\mathfrak{n}$ for some non-zero $\lambda \in \mathbb{R}$. The condition $b(x, x) > 0$ for $x \neq 0$ together with the $*$-invariance of $\mathfrak{n}$ then implies that either $\mathfrak{n} \subset \{x \in \mathfrak{g}^{\mathrm{n.c.}} \mid x^* = x\}$ or $\mathfrak{n} \subset \{x \in \mathfrak{g}^{\mathrm{n.c.}} \mid x^* = -x\}$. Since $\mathfrak{g}^{\mathrm{n.c.}}$ is non-compact, the second possibility is fulfilled. Hence $\mathfrak{g} := \mathfrak{k} \oplus i\mathfrak{n}$ is a L^*-algebra of compact type, which is simple. Let G be the connected L^*-group generated by the exponentials of operator in $\mathfrak{g}$. The affine coadjoint orbit of G defined by the derivation $[D, .]$ is infinite-dimensional irreducible and Hermitian-symmetric, hence is isomorphic to one of the affine adjoint orbits listed in Theorem 1.1. The corollary then follows by duality (see Examples 3.12-3.16). □

□ **Proof of Corollary** 1.2 :
By Theorem (3.9) and the discussion after Theorem (4.2) in [11], every Hermitian-symmetric space of compact (resp. non-compact) type is isomorphic to the orthogonal product of irreducible Hermitian-symmetric spaces of compact (resp. non-compact) type. The irreducible pieces are of type I-VI and described in paragraph 3 in [10]. The types V and VI correspond to the exceptional Lie groups E6

and E7, which are of finite dimension. By the finite-dimensional theory, finite-dimensional Hermitian-symmetric spaces are coadjoint orbits of their groups of isometries (see e.g. Theorem 8.89 in [4]). An infinite-dimensional irreducible Hermitian-symmetric space is of type I, II, III or IV, and is isomorphic (see paragraph 3 in [10]) to one of the affine coadjoint orbits listed in Theorem 1.1 or Corollary 3.17. Both the restricted Grassmannian and the Grassmannian of p-dimensional subspaces of a separable Hilbert space, with $p < +\infty$, are Hermitian-symmetric spaces of type I. Now the theorem follows by taking the product of the L^*-groups acting on each irreducible pieces.

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