THE RESTRICTED GRASSMANNIAN, BANACH LIE-POISSON SPACES, AND COADJOINT ORBITS

DANIEL BELTIŢĂ, TUDOR S. RATIU, AND ALICE BARBARA TUMPACH

ABSTRACT. We investigate some basic questions concerning the relationship between the restricted Grassmannian and the theory of Banach Lie-Poisson spaces. By using universal central extensions of Lie algebras, we find that the restricted Grassmannian is symplectomorphic to symplectic leaves in certain Banach Lie-Poisson spaces, and the underlying Banach space can be chosen to be even a Hilbert space. Smoothness of numerous adjoint and coadjoint orbits of the restricted unitary group is also established. Several pathological properties of the restricted algebra are pointed out.
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1. INTRODUCTION

The present paper is devoted to an investigation of the relationship between the restricted Grassmannian and the recently initiated theory of Banach Lie-Poisson spaces.

The restricted Grassmannian (whose definition is recalled after Proposition 2.11 below) is a quite remarkable infinite-dimensional *Kähler* manifold that plays an important role in many areas of mathematics and physics. There are many interesting objects related to the restricted Grassmannian, such as: loop groups (see Proposition 8.3.3 in [PS90]), the coadjoint orbits $\mathrm{Diff}^+(S^1)/S^1$ and $\mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1)$ of the group of orientation-preserving diffeomorphisms of the circle (Proposition 6.8.2 in [PS90] and Proposition 5.3 in [Se81]). It is related to the integrable system defined by the KP hierarchy (see [SW85]) and to the fermionic second quantization (see [Wu01]). On the other hand, the notion of a Banach Lie-Poisson space was recently introduced in [OR03] and is an infinite-dimensional version of the Lie-Poisson spaces, that is, the Poisson manifolds provided by dual spaces of finite-dimensional Lie algebras (see for instance [OrR04] for the finite-dimensional theory). This new class of infinite-dimensional linear Poisson manifolds is remarkable in several respects: it includes all the preduals of W^*-algebras, thus establishing a bridge between Poisson geometry and the theory of operator algebras, and hence it provides links with algebraic quantum theories; it interacts in a fruitful way with the theory of extensions of Lie algebras (see [OR04]); and finally, there exist large classes of Banach Lie-Poisson spaces which share with the finite-dimensional Poisson manifolds the fundamental property that the characteristic distribution is integrable, the corresponding integral manifolds being in addition Poisson submanifolds which are symplectic and, in several important situations, are even *Kähler* manifolds (see [BR05]).

We have mentioned here two types of infinite-dimensional Kähler manifolds: the restricted Grassmannians and certain symplectic leaves in infinite-dimensional Lie-Poisson spaces introduced in [OR03]. This brings us to the first question addressed in the present paper:

Question 1.1. *Is the restricted Grassmannian a symplectic leaf in a Banach Lie-Poisson space?*

The main result of our paper is essentially affirmative and the precise answer is given in Section 5. Specifically, we shall employ the method of central extensions to construct a certain Banach Lie-Poisson space $\widetilde{u}_2$ whose characteristic distribution is integrable (Theorem 5.1) and one of the integral manifolds of this distribution is symplectomorphic to the connected component $\mathrm{Gr}^0_{\mathrm{res}}$ of the restricted Grassmannian (Theorem 5.3). Using a similar method, we realize the restricted Grassmannian as a symplectic leaf in yet another Banach Lie-Poisson space, which is the predual to a 1-dimensional central extension of the restricted Lie algebra $\mathfrak{u}_{\mathrm{res}}$. See Section 2 for a detailed discussion of the Poisson geometry of this new Banach Lie-Poisson space $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$.

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This second construction is closely related to another area where the theory of restricted groups interacts with the theory initiated in [OR03]. Specifically, we also address the following question on the predual $(\mathfrak{u}_{\mathrm{res}})_*$ of the restricted Lie algebra:

Question 1.2. *Does the real Banach space $(\mathfrak{u}_{\mathrm{res}})_*$ have a natural structure of Banach Lie-Poisson space and is its characteristic distribution integrable?*

By the very construction of the Banach Lie-Poisson space $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$, the predual $(\mathfrak{u}_{\mathrm{res}})_*$ appears as a Poisson submanifold of $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$ and carries a natural structure of Banach Lie-Poisson space. Nonetheless, the answer to the second part of Question 1.2 turns out to be much more difficult to give than the one to Question 1.1 inasmuch as the restricted algebra $\mathcal{B}_{\mathrm{res}}$ (see Notation 1.3 below) is a dual Banach $*$-algebra with many pathological properties (summarized in Section 6): its unitary group is unbounded, its natural predual is not spanned by its positive cone, and a conjugation theorem for its maximal Abelian $*$-subalgebras fails to be true. Despite these unpleasant properties, we show that the characteristic distribution of $(\mathfrak{u}_{\mathrm{res}})_*$ has numerous smooth integral manifolds, which are, in particular, *smooth* coadjoint orbits of the restricted unitary group $\mathrm{U}_{\mathrm{res}}$ (see Section 3). For the sake of completeness, a short section of the paper (Section 4) is devoted to investigating smoothness of adjoint orbits of $\mathrm{U}_{\mathrm{res}}$.

Notation 1.3. We conclude this Introduction by setting up some notation to be used throughout the paper. In the following, $\mathcal{H}$ will denote a separable complex Hilbert space, endowed with a decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into the orthogonal sum of two closed infinite-dimensional subspaces. The orthogonal projection onto $\mathcal{H}_\pm$ will be denoted by $p_\pm$. The Banach ideal of trace class operators on $\mathcal{H}$ will be denoted by $\mathfrak{S}_1(\mathcal{H})$ and $\mathfrak{S}_2(\mathcal{H})$ will denote the Hilbert ideal of Hilbert-Schmidt operators on $\mathcal{H}$. We let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We shall also need the Banach-Lie group of unitary operators on $\mathcal{H}$,

$$\mathrm{U}(\mathcal{H}) = \{u \in \mathcal{B}(\mathcal{H}) \mid u^* u = u u^* = \mathrm{id}\},$$

whose Lie algebra is

$$\mathfrak{u}(\mathcal{H}) = \{a \in \mathcal{B}(\mathcal{H}) \mid a^* = -a\}.$$

Now let us define the following skew-Hermitian element:

$$d := \mathrm{i}(p_+ - p_-) \in \mathfrak{u}(\mathcal{H}).$$

The restricted Banach algebra and the restricted unitary group are respectively defined as follows:

$$\begin{aligned} \mathcal{B}_{\mathrm{res}} &= \{a \in \mathcal{B}(\mathcal{H}) \mid [d,a] \in \mathfrak{S}_2(\mathcal{H})\} = \{a \in \mathcal{B}(\mathcal{H}) \mid \|a\|_{\mathrm{res}} := \|a\| + \|[d,a]\|_2 < \infty\}, \text{ and} \\ \mathrm{U}_{\mathrm{res}} &= \{u \in \mathrm{U}(\mathcal{H}) \mid [d,u] \in \mathfrak{S}_2(\mathcal{H})\} = \mathrm{U}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{res}}. \end{aligned}$$

The Lie algebra of $\mathrm{U}_{\mathrm{res}}$ is the following Banach Lie algebra:

$$\mathfrak{u}_{\mathrm{res}} = \{a \in \mathfrak{u}(\mathcal{H}) \mid [d,a] \in \mathfrak{S}_2(\mathcal{H})\} = \mathfrak{u}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{res}}.$$

Let us define the following Banach Lie algebra:

$$(\mathfrak{u}_{\mathrm{res}})_* = \{\rho \in \mathfrak{u}(\mathcal{H}) \mid [d,\rho] \in \mathfrak{S}_2(\mathcal{H}),\ p_\pm \rho|_{\mathcal{H}_\pm} \in \mathfrak{S}_1(\mathcal{H}_\pm)\}.$$

A connected Banach Lie group with Lie algebra $(\mathfrak{u}_{\mathrm{res}})_*$ is

$$\mathrm{U}_{1,2} = \{a \in \mathrm{U}(\mathcal{H}) \mid a - \mathrm{id} \in \mathfrak{S}_2(\mathcal{H}),\ p_\pm a|_{\mathcal{H}_\pm} \in \mathrm{id} + \mathfrak{S}_1(\mathcal{H}_\pm)\}.$$

The group U_1 and its Lie algebra $\mathfrak{u}_1$ are defined as follows:

$$\begin{aligned} \mathrm{U}_1 &= \{a \in \mathrm{U}(\mathcal{H}) \mid a - \mathrm{id} \in \mathfrak{S}_1(\mathcal{H})\}, \text{ and} \\ \mathfrak{u}_1 &= \mathfrak{u}(\mathcal{H}) \cap \mathfrak{S}_1(\mathcal{H}). \end{aligned}$$

Finally, the Hilbert-Lie group U_2 and its Lie algebra $\mathfrak{u}_2$ are defined by :

$$\begin{aligned} \mathrm{U}_2 &= \{a \in \mathrm{U}(\mathcal{H}) \mid a - \mathrm{id} \in \mathfrak{S}_2(\mathcal{H})\}, \text{ and} \\ \mathfrak{u}_2 &= \mathfrak{u}(\mathcal{H}) \cap \mathfrak{S}_2(\mathcal{H}). \end{aligned}$$

2. The Banach Lie-Poisson space associated to the universal central extension of $\mathfrak{u}_{\mathrm{res}}$

In this section we construct a Banach Lie-Poisson space $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$ whose dual is the universal central extension of the restricted algebra $\mathfrak{u}_{\mathrm{res}}$. (See [Ne02b] for the definition of universal central extension and Proposition 2.4 below for the justification of this fact.) The Poisson structure of $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$ is defined by (2.8) in Proposition 2.5. Let us first justify the suggestive notation $(\mathfrak{u}_{\mathrm{res}})_*$.

Proposition 2.1. *The Lie algebra $(\mathfrak{u}_{\mathrm{res}})_*$ is a predual of the unitary restricted algebra $\mathfrak{u}_{\mathrm{res}}$, the duality pairing $\langle \cdot\,,\cdot\rangle$ being given by*

$$\langle \cdot\,,\cdot\rangle\ : (\mathfrak{u}_{\mathrm{res}})_*\times \mathfrak{u}_{\mathrm{res}} \to \mathbb{R}, \quad (b,c)\mapsto \mathrm{Tr}\,(bc). \tag{2.1}$$

Proof. Consider two arbitrary elements

$$a=\begin{pmatrix} a_{++} & a_{+-}\\ -a_{+-}^* & a_{--}\end{pmatrix}\in \mathfrak{u}_{\mathrm{res}} \quad\text{and}\quad \rho=\begin{pmatrix} \rho_{++} & -\rho_{-+}^*\\ \rho_{-+} & \rho_{--}\end{pmatrix}\in (\mathfrak{u}_{\mathrm{res}})_*.$$

Then

$$a\rho=\begin{pmatrix} a_{++}\rho_{++}+a_{+-}\rho_{-+} & -a_{++}\rho_{-+}^*+a_{+-}\rho_{--}\\ -a_{+-}^*\rho_{++}+a_{--}\rho_{-+} & a_{+-}^*\rho_{-+}^*+a_{--}\rho_{--}\end{pmatrix}, \tag{2.2}$$

hence

$$\mathrm{Tr}\,(a\rho)=\mathrm{Tr}\,(a_{++}\rho_{++})+2\Re\mathrm{Tr}\,(a_{+-}\rho_{-+})+\mathrm{Tr}\,(a_{--}\rho_{--}), \tag{2.3}$$

where $\Re z$ denotes the real part of the complex number z. Recall that the bilinear functional

$$\mathcal{B}(\mathcal{H}_\pm)\times \mathfrak{S}_1(\mathcal{H}_\pm)\to\mathbb{C},\quad (b,c)\mapsto \mathrm{Tr}\,(bc),$$

induces a topological isomorphism of complex Banach spaces $(\mathfrak{S}_1(\mathcal{H}_\pm))^*\simeq\mathcal{B}(\mathcal{H}_\pm)$. It follows that the trace induces a topological isomorphism of real Banach spaces

$$(\mathfrak{u}(\mathcal{H}_\pm)\cap\mathfrak{S}_1(\mathcal{H}_\pm))^*\simeq \mathfrak{u}(\mathcal{H}_\pm). \tag{2.4}$$

Indeed, the $\mathbb{C}$-linearity of the trace implies that for $b\in\mathcal{B}(\mathcal{H}_\pm)$ the following conditions are equivalent:

$$\big(\forall c\in\mathfrak{u}(\mathcal{H}_\pm)\cap\mathfrak{S}_1(\mathcal{H}_\pm)\big)\quad \mathrm{Tr}\,(bc)=0 \iff \big(\forall c\in\mathfrak{S}_1(\mathcal{H}_\pm)\big)\quad \mathrm{Tr}\,(bc)=0.$$

Moreover the condition

$$\big(\forall c\in\mathfrak{u}(\mathcal{H}_\pm)\cap\mathfrak{S}_1(\mathcal{H}_\pm)\big)\quad \mathrm{Tr}\,(bc)\in\mathbb{R}$$

implies

$$\big(\forall c\in\mathfrak{u}(\mathcal{H}_\pm)\cap\mathfrak{S}_1(\mathcal{H}_\pm)\big)\quad \mathrm{Tr}\,(b+b^*)c=0,$$

hence b belongs to $\mathfrak{u}(\mathcal{H}_\pm)$. On the other hand, the duality pairing of complex Hilbert spaces

$$\mathfrak{S}_2(\mathcal{H}_-,\mathcal{H}_+)\times\mathfrak{S}_2(\mathcal{H}_+,\mathcal{H}_-)\to\mathbb{C},\quad (b,c)\mapsto\mathrm{Tr}\,(bc),$$

induces a duality pairing of the underlying real Hilbert spaces by

$$\mathfrak{S}_2(\mathcal{H}_-,\mathcal{H}_+)\times\mathfrak{S}_2(\mathcal{H}_+,\mathcal{H}_-)\to\mathbb{R},\quad (b,c)\mapsto\Re\,\mathrm{Tr}\,(bc). \tag{2.5}$$

In view of formula (2.3), we conclude that the trace induces a topological isomorphism of real Banach spaces

$$((\mathfrak{u}_{\mathrm{res}})_*)^*\simeq\mathfrak{u}_{\mathrm{res}}.$$

That is, $(\mathfrak{u}_{\mathrm{res}})_*$ is indeed a predual to $\mathfrak{u}_{\mathrm{res}}$, the duality pairing being induced by (2.4) and (2.5). □

Definition 2.2. We define the Banach Lie algebra $\tilde{\mathfrak{u}}_{\mathrm{res}}$ as the central extension of $\mathfrak{u}_{\mathrm{res}}$ with continuous two-cocycle s given by

$$s(A,B):=\mathrm{Tr}\,(A[d,B]), \tag{2.6}$$

for all $A,B\in\mathfrak{u}_{\mathrm{res}}$. That is, $\tilde{\mathfrak{u}}_{\mathrm{res}}$ is the Banach algebra $\mathfrak{u}_{\mathrm{res}}\oplus\mathbb{R}$ endowed with the bracket $[\cdot,\cdot]_d$ defined by

$$[(A,a),(B,b)]_d=([A,B],-s(A,B)). \tag{2.7}$$

Remark 2.3. Note that by the very definition of $\mathfrak{u}_{\rm res}$, one has $[d, \mathfrak{u}_{\rm res}] \subset (\mathfrak{u}_{\rm res})_*$. It follows from the duality pairing (2.1), that s is well-defined by (2.6). To see that s defines a two-cocycle on $\mathfrak{u}_{\rm res}$, let us remark that s is (2i)-times the Schwinger term of [Wu01]. It follows from Corollary II.12 in the aforementioned work that s defines a non-trivial element in the second continuous Lie algebra cohomology space $H^2(\mathfrak{u}_{\rm res}, \mathbb{R})$. The corresponding U(1)-extension of the unitary restricted group $\mathrm{U}_{\rm res}$ is isomorphic to the U(1)-extensions $\mathrm{U}_{\rm res}^{\sim}$ and $\widehat{\mathrm{U}}_{\rm res}$ of $\mathrm{U}_{\rm res}$ constructed in [Wu01].

Proposition 2.4. *The cohomology class $[s]$ is a generator of the continuous Lie algebra cohomology space $H^2(\mathfrak{u}_{\rm res}, \mathbb{R})$.*

Proof. According to Proposition I.11 in [Ne02a], the second continuous Lie algebra cohomology space $H^2(\mathcal{B}_{\rm res}, \mathbb{C})$ of the restricted Lie algebra $\mathcal{B}_{\rm res}$ is 1-dimensional. Note that a continuous $\mathbb{R}$-valued 2-cocycle v on $\mathfrak{u}_{\rm res}$ extends by $\mathbb{C}$-linearity to a continuous $\mathbb{C}$-valued 2-cocycle $v^{\mathbb{C}}$ on the complex Lie algebra $\mathcal{B}_{\rm res}$. The cocycle $v^{\mathbb{C}}$ is a coboundary if and only if there exists a continuous linear map $\alpha\colon \mathcal{B}_{\rm res} \to \mathbb{C}$ such that $v^{\mathbb{C}}(x, y) = \alpha([x, y])$ for every $x, y \in \mathcal{B}_{\rm res}$. But since $v^{\mathbb{C}}$ restricts to the $\mathbb{R}$-valued 2-cocycle v on $\mathfrak{u}_{\rm res}$, this is the case if and only if there exists $\beta := \Re\alpha\colon \mathfrak{u}_{\rm res} \to \mathbb{R}$ such that $v(x, y) = \beta([x, y])$ for every $x, y \in \mathfrak{u}_{\rm res}$. It follows that the extension $v^{\mathbb{C}}$ is a coboundary on $\mathcal{B}_{\rm res}$ if and only if v is a coboundary on $\mathfrak{u}_{\rm res}$. Consequently, there is a natural linear injection of $H^2(\mathfrak{u}_{\rm res}, \mathbb{R})$ into $H^2(\mathcal{B}_{\rm res}, \mathbb{C})$. Since s defines a non-trivial element in $H^2(\mathfrak{u}_{\rm res}, \mathbb{R})$ (see Remark 2.3) and $\dim_{\mathbb{C}} H^2(\mathcal{B}_{\rm res}, \mathbb{C}) = 1$, it follows that $\dim_{\mathbb{R}} H^2(\mathfrak{u}_{\rm res}, \mathbb{R}) = 1$ and thus $H^2(\mathfrak{u}_{\rm res}, \mathbb{R})$ is generated by s. ☐

Proposition 2.5. *The Banach space $(\tilde{\mathfrak{u}}_{\rm res})_*$ is a Banach Lie-Poisson space for the Poisson bracket*

$$\{f, g\}_d(\mu, \gamma) := \langle \mu, [D_\mu f(\mu), D_\mu g(\mu)]\rangle - \gamma s(D_\mu f, D_\mu g) \tag{2.8}$$

where $f, g \in \mathcal{C}^\infty((\tilde{\mathfrak{u}}_{\rm res})_*)$, (μ, γ) is an arbitrary element in $(\tilde{\mathfrak{u}}_{\rm res})_*$, and D_μ denotes the partial Fréchet derivative with respect to $\mu \in (\mathfrak{u}_{\rm res})_*$.

The pairing in equation (2.8) is the duality pairing defined by (2.1). We will denote by $\langle \cdot, \cdot\rangle_d$ the duality pairing between $(\tilde{\mathfrak{u}}_{\rm res})_* = (\mathfrak{u}_{\rm res})_* \oplus \mathbb{R}$ and $\tilde{\mathfrak{u}}_{\rm res} = \mathfrak{u}_{\rm res} \oplus \mathbb{R}$ given by

$$\langle (\mu, \gamma), (A, a)\rangle_d = \langle \mu, A\rangle + \gamma a.$$

Proof of Proposition 2.5. By Theorem 4.2 in [OR03], the Banach space $(\tilde{\mathfrak{u}}_{\rm res})_*$ is a Banach Lie-Poisson space if and only if its dual $\tilde{\mathfrak{u}}_{\rm res}$ is a Banach Lie algebra satisfying $\mathrm{ad}^*_x(\tilde{\mathfrak{u}}_{\rm res})_* \subset (\tilde{\mathfrak{u}}_{\rm res})_* \subset (\tilde{\mathfrak{u}}_{\rm res})^*$ for all $x \in \tilde{\mathfrak{u}}_{\rm res}$. The fact that $\tilde{\mathfrak{u}}_{\rm res}$ is a Banach Lie algebra follows directly from the continuity of s and from the 2-cocycle identity which implies the Jacobi identity of $[\cdot, \cdot]_d$. To see that the coadjoint action of $\tilde{\mathfrak{u}}_{\rm res}$ preserves the predual $(\tilde{\mathfrak{u}}_{\rm res})_*$, note that for every $(A, a), (B, b) \in \tilde{\mathfrak{u}}_{\rm res}$ and every $(\mu, \gamma) \in (\tilde{\mathfrak{u}}_{\rm res})_*$, one has

$$\begin{aligned}\langle -\mathrm{ad}^*_{(A,a)}(\mu, \gamma), (B, b)\rangle_d :=& \langle (\mu, \gamma), \mathrm{ad}_{(A,a)}(B, b)\rangle_d = \langle (\mu, \gamma), [(A, a), (B, b)]_d\rangle_d \\ =& \langle (\mu, \gamma), ([A, B], -s(A, B))\rangle_d = \mathrm{Tr}\, \mu[A, B] - \gamma \mathrm{Tr}\, A[d, B] \\ =& \mathrm{Tr}\, \mu[A, B] + \gamma \mathrm{Tr}\, [d, A]B = \langle (-\mathrm{ad}^*(A)(\mu) + \gamma[d, A], 0), (B, b)\rangle_d.\end{aligned}$$

Since

$$[(\mathfrak{u}_{\rm res})_*, \mathfrak{u}_{\rm res}] \subseteq (\mathfrak{u}_{\rm res})_*, \tag{2.9}$$

and

$$[d, \mathfrak{u}_{\rm res}] \subset (\mathfrak{u}_{\rm res})_*, \tag{2.10}$$

we conclude that $-\mathrm{ad}^*(A)(\mu) + \gamma[d, A]$ belongs to $(\mathfrak{u}_{\rm res})_*$ for every $A \in \mathfrak{u}_{\rm res}$. Hence the predual $(\tilde{\mathfrak{u}}_{\rm res})_*$ is preserved by the coadjoint action. Referring again to Theorem 4.2 in [OR03], it follows that the Poisson bracket of f, $g \in \mathcal{C}^\infty((\tilde{\mathfrak{u}}_{\rm res})_*)$ is given by

$$\{f, g\}_d(\mu, \gamma) = \langle (\mu, \gamma), [Df(\mu, \gamma), Dg(\mu, \gamma)]\rangle_d.$$

Denoting respectively by D_μ and D_γ the partial Fréchet derivatives with respect to $\mu \in (\mathfrak{u}_{\rm res})_*$ and $\gamma \in \mathbb{R}$, one has

$$\begin{aligned}\{f, g\}_d(\mu, \gamma) &= \langle (\mu, \gamma), [(D_\mu f, D_\gamma f), (D_\mu g, D_\gamma g)]_d\rangle_d \\ &= \langle (\mu, \gamma), ([D_\mu f, D_\mu g], -s(D_\mu f, D_\mu g))\rangle_d \\ &= \langle \mu, [D_\mu f, D_\mu g]\rangle - \gamma s(D_\mu f, D_\mu g),\end{aligned}$$

and this ends the proof. ☐

Remark 2.6. By Theorem 4.2 in [OR03], it follows that the Hamiltonian vector field associated to a smooth function h on $(\mathfrak{u}_{\rm res})_*$ is given by

$$X_h(\mu,\gamma) = -\mathrm{ad}^*_{(D_\mu h, D_\gamma h)}(\mu,\gamma) = \big(-\mathrm{ad}^*_{D_\mu h}\mu - \gamma[D_\mu h, d], 0\big). \tag{2.11}$$

Remark 2.7. Note that, for each $\gamma \in \mathbb{R}$, $(\mathfrak{u}_{\rm res})_* \oplus \{\gamma\}$ is a Poisson submanifold of $(\tilde{\mathfrak{u}}_{\rm res})_*$ for the following Poisson bracket on the first factor

$$\{f,g\}_{d,\gamma}(\mu) := \langle \mu, [D_\mu f(\mu), D_\mu g(\mu)]\rangle - \gamma s(D_\mu f, D_\mu g).$$

Remark 2.8. The central extension $(\tilde{\mathfrak{u}}_{\rm res})_*$ of the Banach Lie-Poisson space $(\mathfrak{u}_{\rm res})_*$ is a particular example of the extensions of Banach Lie-Poisson spaces constructed in [OR04]. Indeed formula (2.8) for the bracket of two functions on $(\tilde{\mathfrak{u}}_{\rm res})_*$ can be alternatively deduced from the general formula (5.6) in Theorem 5.2 of [OR04], with $\mathfrak{c} = \mathbb{R}$, $\mathfrak{a} = (\mathfrak{u}_{\rm res})_*$, $\varphi = 0$ and $\omega = -s$. The pairing in the second term of the right hand side of (5.6), Theorem 5.2, [OR04], is, in this special case, just the pairing between the real line and its dual given by multiplication of real numbers (the element $c \in \mathfrak{c}$ is γ), and the bracket of partial derivatives of the functions f and g with respect to c vanishes since $\mathbb{R}$ is commutative.

Proposition 2.9. *The unitary group* $\mathrm{U}_{\rm res}$ *acts on the Poisson manifold* $(\mathfrak{u}_{\rm res})_* \oplus \{\gamma\} \subset (\tilde{\mathfrak{u}}_{\rm res})_*$ *by affine coadjoint action as follows. For* $g \in \mathrm{U}_{\rm res}$,

$$g \cdot (\mu,\gamma) := \big(\mathrm{Ad}^*(g^{-1})(\mu) - \gamma\sigma(g), \gamma\big)$$

where $\mu \in (\mathfrak{u}_{\rm res})_*$, $\gamma \in \mathbb{R}$, *and where*

$$\begin{aligned}\sigma\colon \mathrm{U}_{\rm res} &\to (\mathfrak{u}_{\rm res})_*,\\ g &\mapsto gdg^{-1} - d.\end{aligned}$$

Proof. Let us verify that for every $g \in \mathrm{U}_{\rm res}$ we have $g\,d\,g^{-1} - d \in (\mathfrak{u}_{\rm res})_*$. Consider the block decomposition of g with respect to the direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

$$g = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--}\end{pmatrix} \in \mathrm{U}_{\rm res}.$$

One has

$$\begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--}\end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i}\end{pmatrix}\begin{pmatrix} g^*_{++} & g^*_{-+} \\ g^*_{+-} & g^*_{--}\end{pmatrix} = \begin{pmatrix} \mathrm{i}g_{++}g^*_{++} - \mathrm{i}g_{+-}g^*_{+-} & \mathrm{i}g_{++}g^*_{-+} - \mathrm{i}g_{+-}g^*_{--} \\ \mathrm{i}g_{-+}g^*_{++} - \mathrm{i}g_{--}g^*_{+-} & \mathrm{i}g_{-+}g^*_{-+} - \mathrm{i}g_{--}g^*_{--}.\end{pmatrix} \tag{2.12}$$

Since $g_{\pm\mp}$ belongs to $\mathfrak{S}_2(\mathcal{H}_\mp, \mathcal{H}_\pm)$, the off-diagonal blocks of the right hand side are in $\mathfrak{S}_2(\mathcal{H}_\pm, \mathcal{H}_\mp)$. Further, since

$$\begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--}\end{pmatrix}\begin{pmatrix} g^*_{++} & g^*_{-+} \\ g^*_{+-} & g^*_{--}\end{pmatrix} = \begin{pmatrix} g_{++}g^*_{++} + g_{+-}g^*_{+-} & g_{++}g^*_{-+} + g_{+-}g^*_{--} \\ g_{-+}g^*_{++} + g_{--}g^*_{+-} & g_{-+}g^*_{-+} + g_{--}g^*_{--}\end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id}\end{pmatrix},$$

and since $\mathfrak{S}_2 \cdot \mathfrak{S}_2 \subset \mathfrak{S}_1$, one has

$$g_{++}g^*_{++} = \mathrm{id} - g_{+-}g^*_{+-} \in \mathrm{id} + \mathfrak{S}_1(\mathcal{H}_+)$$

and

$$g_{--}g^*_{--} = \mathrm{id} - g_{-+}g^*_{-+} \in \mathrm{id} + \mathfrak{S}_1(\mathcal{H}_-).$$

Consequently,

$$g_{++}g^*_{++} - g_{+-}g^*_{+-} \in \mathrm{id} + \mathfrak{S}_1(\mathcal{H}_+)$$

and

$$g_{-+}g^*_{-+} - g_{--}g^*_{--} \in -\mathrm{id} + \mathfrak{S}_1(\mathcal{H}_-).$$

Moreover, it is clear that the result of the multiplication (2.12) is skew-symmetric. Hence for all $g \in \mathrm{U}_{\rm res}$ we have $g\,d\,g^{-1} - d \in (\mathfrak{u}_{\rm res})_*$.

Denoting by $\mathrm{Aff}\,((\mathfrak{u}_{\rm res})_*)$ the affine group of transformations of $(\mathfrak{u}_{\rm res})_*$, it remains to show that

$$\begin{aligned}(\mathrm{Ad}^*, \gamma\sigma)\colon \mathrm{U}_{\rm res} &\to \mathrm{Aff}((\mathfrak{u}_{\rm res})_*) = \mathrm{GL}((\mathfrak{u}_{\rm res})_*) \rtimes (\mathfrak{u}_{\rm res})_*\\ g &\mapsto (\mathrm{Ad}^*(g^{-1}), \gamma\sigma(g))\end{aligned}$$

is a group homomorphism. For this, we have to check that $\gamma\sigma(g_1 g_2) = \mathrm{Ad}^*(g_1^{-1})\gamma\sigma(g_2) + \gamma\sigma(g_1)$ for all g_1, g_2 in $\mathrm{U}_{1,2}$ (see [Ne00]). In fact

$$\begin{aligned}\sigma(g_1 g_2) &= g_1 g_2\, d\, g_2^{-1} g_1^{-1} - d = g_1\left(g_2\, d\, g_2^{-1} - d\right) g_1^{-1} + (g_1\, d\, g_1^{-1} - d)\\ &= \mathrm{Ad}^*(g_1^{-1})\,(\sigma(g_2)) + \sigma(g_1),\end{aligned}$$

and this ends the proof. □

Proposition 2.10. *The isotropy group of* $(0, \gamma) \in (\mathfrak{u}_{\mathrm{res}})_* \oplus \{\gamma\}$ *for the* $\mathrm{U}_{\mathrm{res}}$*-affine coadjoint action is a Lie subgroup of* $\mathrm{U}_{\mathrm{res}}$.

Proof. An element X in the Lie algebra $\mathfrak{u}_{\mathrm{res}}$ of $\mathrm{U}_{\mathrm{res}}$ induces by infinitesimal affine coadjoint action on $(\mathfrak{u}_{\mathrm{res}})_* \oplus \{\gamma\}$ the following vector field

$$\begin{aligned}X\cdot(\mu,\gamma) :=& \frac{d}{dt}\left[\exp(tX)\cdot(\mu,\gamma)\right]_{t=0}\\ =& \Big(\frac{d}{dt}\left[\mathrm{Ad}^*(\exp(-tX))(\mu) - \gamma\sigma(\exp(tX))\right]_{t=0}, 0\Big)\\ =& \left(-\mathrm{ad}^*_X(\mu) - \gamma[X,d], 0\right).\end{aligned}$$

By definition, the Lie algebra of the isotropy group of (μ, γ) is

$$\mathfrak{u}_{(\mu,\gamma)} := \{X \in \mathfrak{u}_{\mathrm{res}} \mid -\mathrm{ad}^*(X)(\mu) + \gamma[X, d] = 0\}$$

The proposition is trivial when μ and γ vanish. For $\mu = 0$ and $\gamma \neq 0$, the Lie algebra $\mathfrak{u}_{(0,\gamma)}$ consist of all elements of $\mathfrak{u}_{\mathrm{res}}$ which commute with d. Hence, for $\gamma \neq 0$, $\mathfrak{u}_{(0,\gamma)} = \mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)$. A topological complement to $\mathfrak{u}_{(0,\gamma)}$ in $\mathfrak{u}_{\mathrm{res}}$ is $\mathfrak{m} := \mathfrak{u}(\mathcal{H}) \cap (\mathfrak{S}_2(\mathcal{H}_+, \mathcal{H}_-) \oplus \mathfrak{S}_2(\mathcal{H}_-, \mathcal{H}_+))$. □

Proposition 2.11. *The smooth affine coadjoint orbits of* $\mathrm{U}_{\mathrm{res}}$ *are tangent to the characteristic distribution of the Poisson manifold* $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$.

Proof. By the proof of Proposition 2.10, the image of the differential of the orbit map is

$$\mathfrak{u}_{\mathrm{res}}\cdot(\mu,\gamma) = \left\{\left(-\mathrm{ad}^*_X(\mu) - \gamma[X,d], 0\right) \mid X \in \mathfrak{u}_{\mathrm{res}}\right\}.$$

By Remark 2.6, the characteristic space at $(\mu, \gamma) \in (\tilde{\mathfrak{u}}_{\mathrm{res}})_*$ is

$$P(\mu,\gamma) = \{X_h(\mu) = \left(-\mathrm{ad}^*_{D_\mu h}\mu - \gamma[D_\mu h, d], 0\right) \mid h \in \mathcal{C}^\infty((\mathfrak{u}_{\mathrm{res}})_*)\} = \{(-\mathrm{ad}^*_X\mu - \gamma[X,d], 0) \mid X \in \mathfrak{u}_{\mathrm{res}}\}.$$

Thus the assertion follows. □

The *restricted Grassmannian* $\mathrm{Gr}_{\mathrm{res}}$ is defined as the set of subspaces W of the Hilbert space $\mathcal{H}$ such that the orthogonal projection from W to $\mathcal{H}_+$ (respectively to $\mathcal{H}_-$) is a Fredholm operator (respectively a Hilbert-Schmidt operator). It follows from Propositions 7.1.2 and 7.1.3 in [PS90] that $\mathrm{Gr}_{\mathrm{res}}$ is a Hilbert manifold and a homogeneous space under the natural action of $\mathrm{U}_{\mathrm{res}}$. According to Proposition II.2 in [Wu01], the connected components of $\mathrm{U}_{\mathrm{res}}$ are the sets

$$\mathrm{U}^k_{\mathrm{res}} = \left\{\begin{pmatrix} U_{++} & U_{+-}\\ U_{-+} & U_{--}\end{pmatrix} \in \mathrm{U}_{\mathrm{res}} \mid \mathrm{index}(U_{++}) = k\right\} \quad \text{for} \quad k \in \mathbb{Z}.$$

The pairwise disjoint sets

$$\mathrm{Gr}^k_{\mathrm{res}} = \{W \in \mathrm{Gr}_{\mathrm{res}} \mid \mathrm{index}(p_+|_W \colon W \to H_+) = k\}, \quad k \in \mathbb{Z}$$

are the images of the connected components of $\mathrm{U}_{\mathrm{res}}$ by the continuous projection $\mathrm{U}_{\mathrm{res}} \to \mathrm{Gr}_{\mathrm{res}} = \mathrm{U}_{\mathrm{res}}/\left(\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)\right)$, and thus they are the connected components of $\mathrm{Gr}_{\mathrm{res}}$. In particular, the connected component of $\mathrm{Gr}_{\mathrm{res}}$ containing $\mathcal{H}_+$ is $\mathrm{Gr}^0_{\mathrm{res}}$. The Kähler structure of the restricted Grassmannian is defined in [PS90], Section 7.8. According to the convention in [PS90], the Kähler form ω_{Gr} of $\mathrm{Gr}_{\mathrm{res}}$ is the $\mathrm{U}_{\mathrm{res}}$-invariant 2-form whose value at $\mathcal{H}_+$ is given by

$$\omega_{\mathrm{Gr}}(X, Y) = 2\Im\mathrm{Tr}\,(X^*Y), \tag{2.13}$$

where $X, Y \in \mathfrak{S}_2(\mathcal{H}_+, \mathcal{H}_-) \simeq T_{\mathcal{H}_+}\mathrm{Gr}_{\mathrm{res}}$ and $\mathfrak{I}z$ denotes the imaginary part of $z \in \mathbb{C}$.. Equivalently, ω_{Gr} is the quotient of the following real-valued anti-symmetric bilinear form Ω_{Gr} on $\mathfrak{u}_{\mathrm{res}}$ which vanishes on $\mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)$ and is invariant under the $\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$-action (see Corollary III.8 in [Wu01]) :

$$\Omega_{\mathrm{Gr}}(A, B) = -2s(A, B) \tag{2.14}$$

where A and B belongs to $\mathfrak{u}_{\mathrm{res}}$. In this correspondence, an element $A = \begin{pmatrix} A_{++} & A_{-+}^* \\ A_{-+} & A_{--} \end{pmatrix}$ in $\mathfrak{u}_{\mathrm{res}}$ is identified with the vector $X = A_{-+}$ in $\mathfrak{S}_2(\mathcal{H}_+, \mathcal{H}_-) \simeq T_{\mathcal{H}_+}(\mathrm{Gr}_{\mathrm{res}})$.

Proposition 2.12. *For every $\gamma \neq 0$, the connected components of the* $\mathrm{U}_{\mathrm{res}}$*-affine coadjoint orbit $\mathcal{O}_{(0,\gamma)}$ of $(0,\gamma) \in (\mathfrak{u}_{\mathrm{res}})_* \oplus \{\gamma\}$ are strong symplectic leaves in the Banach Lie-Poisson space $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$.*

Proof. This follows from the general results given in Theorems 7.3, 7.4 and 7.5 in [OR03], from the above Propositions 2.5, 2.9, 2.10, 2.11, and from the fact that the characteristic subspace $P(0,\gamma)$ is the space $\mathfrak{m} := \mathfrak{u}(\mathcal{H}) \cap (\mathfrak{S}_2(\mathcal{H}_+, \mathcal{H}_-) \oplus \mathfrak{S}_2(\mathcal{H}_-, \mathcal{H}_+))$, which is closed in $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$ (we identify the subspace $\mathfrak{m}$ in $(\mathfrak{u}_{\mathrm{res}})_*$ with the subspace $\mathfrak{m} \oplus \{0\}$ in $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$). ☐

Theorem 2.13. *The connected components of the restricted Grassmannian are strong symplectic leaves in the Banach Lie-Poisson space $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$. More precisely, for every $\gamma \neq 0$, the* $\mathrm{U}_{\mathrm{res}}$*-affine coadjoint orbit $\mathcal{O}_{(0,\gamma)}$ of $(0,\gamma) \in (\mathfrak{u}_{\mathrm{res}})_* \oplus \{\gamma\}$ is isomorphic to the restricted Grassmannian* $\mathrm{Gr}_{\mathrm{res}}$ *via the application*

$$\begin{aligned} \Phi_\gamma \colon \mathrm{Gr}_{\mathrm{res}} &\to \mathcal{O}_{(0,\gamma)} \\ W &\mapsto 2\mathrm{i}\gamma(p_W - p_+), \end{aligned}$$

where p_W denotes the orthogonal projection on W. The pull-back by Φ_γ of the symplectic form on $\mathcal{O}_{(0,\gamma)}$ is $(\gamma/2)$-times the symplectic form ω_{Gr} on $\mathrm{Gr}_{\mathrm{res}}$.

Proof. An element ρ of the affine coadjoint orbit $\mathcal{O}_{(0,\gamma)}$ of $(0,\gamma)$ is of the form

$$\rho = \gamma(g\, d\, g^{-1} - d) = 2\mathrm{i}\gamma(g\, p_+\, g^{-1} - p_+),$$

for some $g \in \mathrm{U}_{\mathrm{res}}$. By Corollary III.4 ii) in [Wu01], Φ_γ is bijective for $\gamma \neq 0$. Since the manifold structure of the orbit $\mathcal{O}_{(0,\gamma)}$ is induced by the identification $\mathcal{O}_{(0,\gamma)} = \mathrm{U}_{\mathrm{res}}/\left(\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)\right)$, it follows from Corollary III.4 i) in [Wu01] that Φ_γ is a diffeomorphism. The symplectic form $\omega_{\mathcal{O}}$ on $\mathcal{O}_{(0,\gamma)}$ is the $\mathrm{U}_{\mathrm{res}}$-invariant symplectic form whose value at $(0,\gamma) \in \mathcal{O}_{(0,\gamma)}$ is the given by

$$\omega_{\mathcal{O}}\,(0,\gamma)\,(X_f(0,\gamma), X_g(0,\gamma)) = \{f\,, g\}_d(0\,,\gamma),$$

where f and g are any smooth function on $(\mathfrak{u}_{\mathrm{res}})_*$. Using formula (2.11) and (2.8), it then follows that

$$\omega_{\mathcal{O}}\,(0,\gamma)\,(\gamma[D_\mu f, d]\,, \gamma[D_\mu g, d]) = -\gamma s(D_\mu f\,, D_\mu g).$$

Hence for every $A, B \in \mathfrak{u}_{\mathrm{res}}$, one has :

$$\omega_{\mathcal{O}}\,(0,\gamma)\,(\gamma[A,d]\,, \gamma[B,d]) = -\gamma s(A\,, B) = \frac{\gamma}{2}\Omega_{\mathrm{Gr}}(A,B).$$

It follows that the real-valued anti-symmetric bilinear form on $\mathfrak{u}_{\mathrm{res}}$ corresponding to the symplectic form $\omega_{\mathcal{O}}$ on $\mathcal{O}_{(0,\gamma)} = \mathrm{U}_{\mathrm{res}}/\left(\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)\right)$ equals $\frac{\gamma}{2}\Omega_{\mathrm{Gr}}$ (where the latter identification is given by the orbit map), and this ends the proof. ☐

Remark 2.14. We refer to the paper [OR04] for additional information on the relationship between the Banach Lie-Poisson spaces and the theory of Lie algebra extensions.

3. Coadjoint orbits of the restricted unitary group

This section includes some partial answers to Question 1.2. The main difficulty is to show that the isotropy group of an element in the predual $(\mathfrak{u}_{\mathrm{res}})_*$ is a Lie subgroup of $\mathrm{U}_{\mathrm{res}}$, or equivalently that its Lie algebra is complemented in $\mathfrak{u}_{\mathrm{res}}$. Using the averaging method developed in [Ba90] and [BP05] for constructing closed complements, we will be able to show that the $\mathrm{U}_{\mathrm{res}}$-coadjoint orbit of every element $\rho \in (\mathfrak{u}_{\mathrm{res}})_*$ which commutes with d is a smooth manifold and that its connected components are symplectic leaves of the characteristic distribution (see Proposition 3.3). It follows that the same conclusion holds for every element $\rho \in (\mathfrak{u}_{\mathrm{res}})_*$ which is $\mathrm{U}_{\mathrm{res}}$-conjugate to an element commuting to d, or equivalently to a diagonal operator with respect to a Hilbert basis compatible with the eigenspaces of d. Nevertheless, the set of elements with the latter property is far from being equal to the whole $(\mathfrak{u}_{\mathrm{res}})_*$. Recall that in finite dimensions, every element in the Lie algebra $\mathfrak{u}(n)$ of the unitary group $\mathrm{U}(n)$ is $\mathrm{U}(n)$-conjugate to a diagonal matrix, or, in other words, $\mathrm{U}(n)$ acts transitively on the set of Cartan subalgebras of $\mathfrak{u}(n)$. This is no longer true in the infinite-dimensional case (see subsection 6.3). It is a difficult question to decide whether a given operator ρ in $(\mathfrak{u}_{\mathrm{res}})_*$ or $\mathfrak{u}_{\mathrm{res}}$ has the good property of being $\mathrm{U}_{\mathrm{res}}$-conjugate to a diagonal operator. In Propositions 3.5 and 3.6, we give some concrete criteria to check that property.

Conjecture 3.1. *The real Banach space* $(\mathfrak{u}_{\rm res})_*$ *has a natural structure of Banach Lie-Poisson space and its characteristic distribution is integrable.*

Remark 3.2. It is clear that

$$(\mathfrak{u}_{\rm res})_* \hookrightarrow \mathfrak{u}_{\rm res}$$

with a continuous inclusion map. On the other hand, it follows at once by the multiplication formula (2.2) that

$$[(\mathfrak{u}_{\rm res})_*, \mathfrak{u}_{\rm res}] \subseteq (\mathfrak{u}_{\rm res})_*, \tag{3.1}$$

which implies that the predual $(\mathfrak{u}_{\rm res})_*$ is left invariant by the coadjoint representation of the Banach Lie algebra $\mathfrak{u}_{\rm res}$. Now the results of [OR03] imply the following two facts:

- The predual Banach space $(\mathfrak{u}_{\rm res})_*$ has a natural structure of Banach Lie-Poisson space.
- If $\rho \in (\mathfrak{u}_{\rm res})_*$ has the property that the corresponding isotropy group

$$\mathrm{U}_{{\rm res},\rho} := \{u \in \mathrm{U}_{\rm res} \mid u\rho u^{-1} = \rho\}$$

  is a Banach Lie subgroup of $\mathrm{U}_{\rm res}$, then the coadjoint orbit $\mathcal{O}_\rho$ is an integral manifold of the characteristic distribution of $(\mathfrak{u}_{\rm res})_*$. Moreover, $\mathcal{O}_\rho$ is a weakly symplectic manifold when equipped with the orbit symplectic structure.

Thus, the desired conclusion will follow as soon as we prove that the isotropy group $\mathrm{U}_{{\rm res},\rho}$ of any $\rho \in (\mathfrak{u}_{\rm res})_*$ is a Banach Lie subgroup of $\mathrm{U}_{\rm res}$.

The Lie algebra of $\mathrm{U}_{{\rm res},\rho}$ is given by

$$\mathfrak{u}_{{\rm res},\rho} = \{a \in \mathfrak{u}_{\rm res} \mid a\rho = \rho a\} = \{a \in \mathfrak{u}_{\rm res} \mid (\forall t \in \mathbb{R}) \quad \alpha_t(a) = a\},$$

where

$$\alpha\colon \mathbb{R} \to \mathcal{B}(\mathfrak{u}_{\rm res}), \quad \alpha(t)b := \alpha_t(b) := \exp(t\rho)\cdot b \cdot \exp(-t\rho).$$

It is clear that α is a group homomorphism. Moreover, since $\rho \in (\mathfrak{u}_{\rm res})_* \subseteq \mathfrak{u}_{\rm res}$ and the adjoint action of the Banach Lie group $\mathrm{U}_{\rm res}$ is continuous, it follows that $\alpha\colon \mathbb{R} \to \mathcal{B}(\mathfrak{u}_{\rm res})$ is norm continuous.

On the other hand, it follows by (3.1) that

$$(\forall t \in \mathbb{R}) \quad \alpha_t((\mathfrak{u}_{\rm res})_*) \subseteq (\mathfrak{u}_{\rm res})_*, \tag{3.2}$$

since $\rho \in (\mathfrak{u}_{\rm res})_*$. Then the concrete form of the duality pairing between $(\mathfrak{u}_{\rm res})_*$ and $\mathfrak{u}_{\rm res}$ (see (2.3)) shows that

$$(\forall t \in \mathbb{R}) \quad (\alpha_t|_{(\mathfrak{u}_{\rm res})_*})^* = \alpha_{-t}, \tag{3.3}$$

and in particular each operator $\alpha_t\colon \mathfrak{u}_{\rm res} \to \mathfrak{u}_{\rm res}$ is weak* continuous.

Now a complement to $\mathfrak{u}_{{\rm res},\rho}$ in $\mathfrak{u}_{\rm res}$ can be constructed by the averaging technique over the amenable group $(\mathbb{R}, +)$ provided one has $\sup\limits_{t\in\mathbb{R}} \|\alpha_t\| < \infty$. (Some references for the aforementioned averaging technique are [Ba90], the proof of Proposition 3.4 in [BR05], and [BP05].)

Additionally we note that since for every operator $T : X \to Y$ between the Banach spaces X and Y the norm of T equals the norm of its dual T^*, it is enough to estimate uniformly the norm of α_t restricted to the predual $(\mathfrak{u}_{\rm res})_*$. This restriction is an adjoint action of the group corresponding to the predual.

Proposition 3.3. *If* $\rho \in (\mathfrak{u}_{\rm res})_*$ *and* $[d, \rho] = 0$*, then the coadjoint isotropy group of* ρ *is a Banach Lie subgroup of* $\mathrm{U}_{\rm res}$ *and the connected components of the corresponding* $\mathrm{U}_{\rm res}$*-coadjoint orbit* $\mathcal{O}_\rho$ *are smooth leaves of the characteristic distribution of* $(\mathfrak{u}_{\rm res})_*$.

Proof. According to Remark 3.2 it suffices to show that $\sup\limits_{t\in\mathbb{R}} \|\alpha_t\| < \infty$. The hypothesis $[d, \rho] = 0$ shows that ρ preserves $\mathcal{H}_+$ and $\mathcal{H}_-$, that is

$$\rho = \begin{pmatrix} \rho_{++} & 0 \\ 0 & \rho_{--} \end{pmatrix} \in (\mathfrak{u}_{\rm res})_*.$$

An element $b \in (\mathfrak{u}_{\rm res})_*$ with block decomposition with respect to the direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

$$b = \begin{pmatrix} b_{++} & b_{+-} \\ b_{-+} & b_{--} \end{pmatrix}$$

is the sum of an element

$$b_1 = \begin{pmatrix} b_{++} & 0 \\ 0 & b_{--} \end{pmatrix}$$

in the Lie algebra $\mathfrak{u}_0 := \mathfrak{u}_1 \cap (\mathfrak{u}(\mathcal{H}_+) \times \mathfrak{u}(\mathcal{H}_-))$ and an element

$$b_2 = \begin{pmatrix} 0 & b_{+-} \\ b_{-+} & 0 \end{pmatrix}$$

in the topological complement $\mathfrak{m} = \mathfrak{u}(\mathcal{H}) \cap (\mathfrak{S}_2(\mathcal{H}_+, \mathcal{H}_-) \oplus \mathfrak{S}_2(\mathcal{H}_-, \mathcal{H}_+))$ of $\mathfrak{u}_0$ in $(\mathfrak{u}_{\rm res})_*$. Accordingly,

$$\begin{aligned} \|\alpha_t(b)\|_{(\mathfrak{u}_{\rm res})_*} &= \| \exp(t\rho) b \exp(-t\rho)\|_{(\mathfrak{u}_{\rm res})_*} \\ &= \| \exp(t\rho) b_1 \exp(-t\rho) + \exp(t\rho) b_2 \exp(-t\rho)\|_{(\mathfrak{u}_{\rm res})_*} \\ &= \|e^{{\rm ad}(t\rho)}(b_1) + e^{{\rm ad}(t\rho)}(b_2)\|_{(\mathfrak{u}_{\rm res})_*}. \end{aligned}$$

Since ${\rm ad}(t\rho)$ preserves both $\mathfrak{u}_0$ and $\mathfrak{m}$, it follows that

$$e^{{\rm ad}(t\rho)}(b_1) \in \mathfrak{u}_0 \quad \text{and} \quad e^{{\rm ad}(t\rho)}(b_2) \in \mathfrak{m}.$$

By the very definition of the norm $\| \cdot \|_{(\mathfrak{u}_{\rm res})_*}$, one has

$$\|\alpha_t(b)\|_{(\mathfrak{u}_{\rm res})_*} = \|e^{{\rm ad}(t\rho)}(b_1)\|_1 + \|e^{{\rm ad}(t\rho)}(b_2)\|_2,$$

where $\| \cdot \|_1$ (respectively $\| \cdot \|_2$) is the usual norm in $\mathfrak{S}_1$ (respectively $\mathfrak{S}_2$). Since the conjugation by a unitary element preserves both $\|\cdot\|_1$ and $\|\cdot\|_2$, it follows that α_t acts by isometries on $(\mathfrak{u}_{\rm res})_*$, in particular $\sup\limits_{t\in\mathbb{R}} \|\alpha_t\| < \infty$. □

Remark 3.4. The calculation in the proof of Proposition 3.3 actually shows that for every $u \in {\rm U}_{\rm res}$ satisfying $[d, u] = 0$ we have $\|ubu^{-1}\|_{\rm res} = \|b\|_{\rm res}$ whenever $b \in \mathcal{B}_{\rm res}$. In fact

$$\|ubu^{-1}\|_{\rm res} = \|ubu^{-1}\| + \|[d, ubu^{-1}]\|_2 = \|b\| + \|u[d, b]u^{-1}\|_2 = \|b\| + \|[d, b]\|_2 = \|b\|_{\rm res}$$

where the second equality follows since $[d, u] = 0$.

Corollary 3.5. *If $\rho \in (\mathfrak{u}_{\rm res})_*$ is a finite-rank operator, then the coadjoint isotropy group of ρ is a Banach Lie subgroup of ${\rm U}_{\rm res}$ and the corresponding ${\rm U}_{\rm res}$-coadjoint orbit $\mathcal{O}_\rho$ is a smooth leaf of the characteristic distribution of $(\mathfrak{u}_{\rm res})_*$.*

Proof. The set of finite-rank operators $\mathcal{F}$ is a dense subset of the predual $(\mathfrak{u}_{\rm res})_*$. For every skew-symmetric finite-rank operator F there exists a unitary operator $u \in \mathbf{1} + \mathcal{F}$, such that uFu^{-1} leaves both $\mathcal{H}_-$ and $\mathcal{H}_+$ invariant. (This follows since any two finite-rank operators are contained in a certain finite-dimensional Lie algebra of finite-rank operators; see for instance Lemma 1 in Chapter I of [dlH72] or Proposition 3.1 in [St75].) Note that $u \in {\rm U}_{\rm res}$, and the isotropy groups of the elements F and uFu^{-1} are conjugated by the element u. Hence the isotropy group at any finite-rank operator is a Banach-Lie subgroup of ${\rm U}_{\rm res}$, and this shows that the conclusion of Proposition 3.3 is satisfied if we replace the hypothesis $[d, \rho] = 0$ by the condition that ρ is a finite-rank operator. □

Corollary 3.6. *Assume that $\rho \in (\mathfrak{u}_{\rm res})_*$ and that there exist an orthonormal basis $\{e_n\}_{n\geq1}$ and the real numbers $t \in (0, 1)$ and $s \in (0, 3(1 - t)/100]$ such that the following conditions are satisfied:*

(i) *We have $\{e_n \mid n \geq 1\} \subseteq \mathcal{H}_+ \cup \mathcal{H}_-$.*
(ii) *The matrix $(\rho_{mn})_{m,n\geq1}$ of ρ with respect to the basis $\{e_n\}_{n\geq1}$ has the properties*

$$|\rho_{m+1,n+1}| \leq t|\rho_{m,n}| \text{ whenever } m, n \geq 1,$$

and

$$|\rho_{m,n}|^2 \leq \frac{s^2}{(mn)^2}|\rho_{mm}\rho_{nn}| \text{ whenever } m, n \geq 1 \text{ and } m \neq n.$$

Then the coadjoint isotropy group of ρ is a Banach-Lie subgroup of ${\rm U}_{\rm res}$ and the corresponding ${\rm U}_{\rm res}$-coadjoint orbit $\mathcal{O}_\rho$ is a smooth leaf of the characteristic distribution of $(\mathfrak{u}_{\rm res})_*$.

Proof. It follows at once by Theorem 1 in [Hk85] that there exists an operator $a = -a^* \in \mathfrak{S}_2(\mathcal{H})$ such that the operator $u\rho u^{-1}$ is diagonal with respect to the basis $\{e_n\}_{n\geq1}$, where $u = \exp a$. In particular we have $u \in {\rm U}_2 \subseteq {\rm U}_{\rm res}$ and $[d, u\rho u^{-1}] = 0$, so that we can use Proposition 3.3 to get the desired conclusion. □

Remark 3.7. Let $\rho \in \mathcal{B}(\mathcal{H})$. In addition to the applications of Proposition 3.3 in the proofs of Corollaries 3.5 and 3.6, we note that each of the following two conditions is equivalent to the existence of an unitary operator $u \in \mathrm{U}_{\mathrm{res}}$ such that $[d, u\rho u^{-1}] = 0$:

(i) There exists $p \in \mathcal{B}(\mathcal{H})$ such that $p = p^* = p^2$, $p - p_+ \in \mathfrak{S}_2(\mathcal{H})$, and $\rho p = p\rho$.

(ii) There exists an element $\mathcal{W} \in \mathrm{Gr}_{\mathrm{res}}$ such that $\rho(\mathcal{W}) \subseteq \mathcal{W}$.

In fact, our assertion concerning (i) follows at once since

$$\{p \in \mathcal{B}(\mathcal{H}) \mid p = p^* = p^2 \text{ and } p - p_+ \in \mathfrak{S}_2(\mathcal{H})\} = \{up_+u^{-1} \mid u \in \mathrm{U}_{\mathrm{res}}\}$$

according to Lemma 3.1 in [Ca85].

On the other hand, the assertion on condition (ii) holds since by Proposition 7.1.3 in [PS90] we have

$$\mathrm{Gr}_{\mathrm{res}} = \{u(\mathcal{H}_+) \mid u \in \mathrm{U}_{\mathrm{res}}\}$$

and, in addition, if $p \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection onto some closed subspace $\mathcal{W} \subseteq \mathcal{H}$ then $\rho(\mathcal{W}) \subseteq \mathcal{W}$ if and only if $[p, \rho] = 0$.

4. Some smooth adjoint orbits of the restricted unitary group

For the sake of completeness, we are going to investigate in this section the smoothness of adjoint orbits of the restricted unitary group. In particular, we shall find sufficiently many smooth adjoint orbits of $\mathrm{U}_{\mathrm{res}}$ to fill an open subset of the Lie algebra $\mathfrak{u}_{\mathrm{res}}$ (Proposition 4.2 below).

Lemma 4.1. *Assume that the element*

$$\rho = \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix} \in \mathfrak{u}_{\mathrm{res}}$$

satisfies the conditions

$$\sigma(\rho_{++}) \cap \sigma(\rho_{--}) = \varnothing, \tag{4.1}$$

and

$$\|\rho_{+-}\|_2 < \frac{1}{2}\mathrm{dist}(\sigma(\rho_{++}), \sigma(\rho_{--})). \tag{4.2}$$

Then there exists $u \in \mathrm{U}_{\mathrm{res}}$ such that $[d, u^{-1}\rho u] = 0$.

Proof. The hypotheses (4.1) and (4.2) imply that there exists a Hilbert-Schmidt operator $k\colon \mathcal{H}_+ \to \mathcal{H}_-$ satisfying the operator Riccati equation

$$k\rho_{+-}k + k\rho_{++} - \rho_{--}k = \rho_{-+}.$$

(This result was obtained in [Mo95]; see also Theorem 4.6 and Remark 4.7 in [ALT01], as well as [AMM03].) Then the operator

$$g = \begin{pmatrix} \mathrm{id}_{\mathcal{H}_+} & k^* \\ k & -\mathrm{id}_{\mathcal{H}_-} \end{pmatrix}$$

is invertible and has the properties $[d, g] \in \mathfrak{S}_2(\mathcal{H})$, $g = g^*$, $[d, g^2] = 0$ and

$$[d, g^{-1}\rho g] = 0 \tag{4.3}$$

(see Subsection 2.3 in [ALT01]). Now let $g = us$ be the polar decomposition of the invertible operator $g \in \mathcal{B}(\mathcal{H})$, where $u \in \mathcal{B}(\mathcal{H})$ is unitary and $s = (g^*g)^{1/2}$.

On the other hand, since $d^* = -d$, it follows that the commutant $\{d\}'$ is a von Neumann algebra of operators on $\mathcal{H}$. Thus, since $g = g^*$ and $g^*g = g^2 \in \{d\}'$, it is straightforward to deduce that $(g^*g)^{1/2} \in \{d\}'$, that is, $[d, s] = 0$. Now recall that $[d, g] \in \mathfrak{S}_2(\mathcal{H})$ to deduce that the unitary operator $u = gs^{-1}$ satisfies $[d, u] \in \mathfrak{S}_2(\mathcal{H})$, that is, $u \in \mathrm{U}_{\mathrm{res}}$.

Moreover by (4.3) we have

$$0 = [d, g^{-1}\rho g] = [d, s^{-1}u^{-1}\rho us] = s^{-1}[d, u^{-1}\rho u]s,$$

where the latter equality follows since we have seen that $[d, s] = 0$. Now we get $[d, u^{-1}\rho u] = 0$, as desired. □

Proposition 4.2. *There exists an open $\mathrm{U}_{\mathrm{res}}$-invariant neighborhood V of $d \in \mathfrak{u}_{\mathrm{res}}$ such that V is a union of smooth adjoint orbits of the Banach Lie group $\mathrm{U}_{\mathrm{res}}$.*

Proof. Denote by V_0 the set of all elements
$$\rho = \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix} \in \mathfrak{u}_{\rm res}$$
satisfying conditions
$$\sigma(\rho_{\pm\pm}) \subseteq \{y \in \mathrm{i}\mathbb{R} \mid |y \mp \mathrm{i}| < 1/3\}$$
and
$$\|\rho_{\pm}\|_2 < \frac{2}{3}.$$
It is clear that V_0 is an open neighborhood of $d \in \mathfrak{u}_{\rm res}$. We are going to show that the set
$$V := \bigcup_{u \in \mathrm{U}_{\rm res}} \mathrm{Ad}_{\mathrm{U}_{\rm res}}(u) V_0 \subseteq \mathfrak{u}_{\rm res}$$
has the desired properties.

Indeed, V is clearly invariant under the adjoint action of $\mathrm{U}_{\rm res}$, it is a union of open sets, and one of these open sets contains d. Moreover, it follows by Lemma 4.1 along with the construction of V that for every $\rho \in V$ there exists $u \in \mathrm{U}_{\rm res}$ such that $[d, u^{-1}\rho u] = 0$. Next denote $\tilde\rho = u^{-1}\rho u$, so that $\exp(t\rho) = u\exp(t\tilde\rho)u^{-1}$ for all $t \in \mathbb{R}$. Then for all $t \in \mathbb{R}$ and $b \in \mathfrak{u}_{\rm res}$ it follows by means of Remark 3.4 that
$$\begin{aligned}\|\exp(t\rho)\cdot b \cdot \exp(-t\rho)\|_{\rm res} &= \|u\exp(t\tilde\rho)\cdot u^{-1}\cdot b\cdot u\cdot \exp(-t\tilde\rho)\cdot u^{-1}\|_{\rm res} \\ &\le \|u\|_{\rm res}\cdot\|\exp(t\tilde\rho)\cdot u^{-1}\cdot b\cdot u\cdot\exp(-t\tilde\rho)\|_{\rm res}\cdot\|u^{-1}\|_{\rm res} \\ &= \|u\|_{\rm res}\cdot\|u^{-1}bu\|_{\rm res}\cdot\|u^{-1}\|_{\rm res} \\ &\le \|u\|^2_{\rm res}\cdot\|u^{-1}\|^2_{\rm res}\cdot\|b\|_{\rm res}.\end{aligned}$$
Consequently the 1-parameter group
$$\alpha\colon \mathbb{R} \to \mathcal{B}(\mathfrak{u}_{\rm res}), \quad \alpha_t(b) = \exp(t\rho)\cdot b\cdot\exp(-t\rho)$$
satisfies
$$\sup_{t\in\mathbb{R}} \|\alpha_t\| \le \|u\|^2_{\rm res}\cdot\|u^{-1}\|^2_{\rm res}.$$
Now the arguments in Remark 3.2 show that the adjoint isotropy group of ρ is a Lie subgroup of $\mathrm{U}_{\rm res}$, and thus the adjoint orbit of ρ is smooth. □

Corollary 4.3. *There exists an open* $\mathrm{U}_{1,2}$*-invariant open neighborhood* V *of* $d \in \mathfrak{u}_{\rm res} = \mathfrak{u}_{1,2}^*$ *such that* V *is a union of smooth coadjoint orbits of the Banach-Lie group* $\mathrm{U}_{1,2}$.

Proof. Apply Proposition 4.2 along with the fact that $\mathrm{U}_{1,2} \hookrightarrow \mathrm{U}_{\rm res}$ and the adjoint action of $\mathrm{U}_{\rm res}$ restricts to the coadjoint action of $\mathrm{U}_{1,2}$. □

5. The Banach Lie-Poisson space associated to the central extension of $\mathfrak{u}_2$

Denote by $\tilde{\mathfrak{u}}_2 := \mathfrak{u}_2 \oplus \mathbb{R}$ the central extension of $\mathfrak{u}_2$ defined by the restriction of s to $\mathfrak{u}_2 \times \mathfrak{u}_2$, where s is the two-cocycle defined in (2.6). The natural isomorphism $(\tilde{\mathfrak{u}}_2)^* \simeq \tilde{\mathfrak{u}}_2$ implies that $\tilde{\mathfrak{u}}_2$ is a Banach Lie-Poisson space, for the Poisson bracket given by
$$\{f, g\}_d(\mu, \gamma) := \langle \mu, [D_\mu f(\mu), D_\mu g(\mu)]\rangle - \gamma s(D_\mu f, D_\mu g)$$
where $f, g \in \mathcal{C}^\infty(\tilde{\mathfrak{u}}_2)$, (μ, γ) is an arbitrary element in $\tilde{\mathfrak{u}}_2$, and D_μ denotes the partial Fréchet derivative with respect to $\mu \in \mathfrak{u}_2$.

Theorem 5.1. *The characteristic distribution of the Banach Lie-Poisson space* $\tilde{\mathfrak{u}}_2$ *is integrable.*

Proof. In order to prove that the characteristic distribution is integrable, it suffices to check that all of the affine coadjoint isotropy groups are Lie subgroups of the Hilbert Lie group U_2. For this purpose we note that, for arbitrary $(\mu, \gamma) \in \tilde{\mathfrak{u}}_2$, the corresponding isotropy group of the affine coadjoint action of U_2 on $\tilde{\mathfrak{u}}_2$ is
$$(\mathrm{U}_2)_{(\mu,\gamma)} = \{g \in \mathrm{U}_2 \mid \mu = g\mu g^{-1} - \gamma g d g^{-1} + \gamma d\},$$
according to the explicit expression of the affine coadjoint action in Proposition 2.9. The previous equality implies that
$$(\mathrm{U}_2)_{(\mu,\gamma)} = \{g \in \mathbb{C}\mathbf{1} + \mathfrak{S}_2(\mathcal{H}) \mid g^* g = g g^* = 1 \text{ and } \mu = g\mu g^{-1} - \gamma g d g^{-1} + \gamma d\},$$

and now it is clear that $(\mathrm{U}_2)_{(\mu,\gamma)}$ is an algebraic subgroup of degree ≤ 2 of the group of invertible elements in the unital Banach algebra $\mathbb{C}\mathbf{1}+\mathfrak{S}_2(\mathcal{H})$. Then the Harris-Kaup theorem (see for instance Theorem 4.13 in [Be06]) implies that $(\mathrm{U}_2)_{(\mu,\gamma)}$ is a Lie group with respect to the topology inherited from $\mathbb{C}\mathbf{1}+\mathfrak{S}_2(\mathcal{H})$. In particular, this topology coincides with the one inherited from U_2. Since U_2 is a Hilbert Lie group, hence the Lie algebra of $(\mathrm{U}_2)_{(\mu,\gamma)}$ has a complement in the Lie algebra of U_2, it then follows that $(\mathrm{U}_2)_{(\mu,\gamma)}$ is a Banach Lie subgroup of U_2, and this concludes the proof. (Compare Remark 3.2.) □

The transitivity of the action of the Lie group U_2 on the connected component $\mathrm{Gr}^0_{\mathrm{res}}$ of the restricted Grassmannian has been established in Theorem 3.5 in [Ca85], and Proposition V.7 in [Ne02a]. That the action of the subgroup $\mathrm{U}_{1,2}$ of U_2 on $\mathrm{Gr}^0_{\mathrm{res}}$ is transitive has been proved in section 1.3.4 of [Tu05] with the help of the canonical basis defined in section 7.3 of [PS90] and associated to any element of the restricted Grassmannian. Below we give a shorter and geometrical proof of the latter fact.

Proposition 5.2. *The connected component* $\mathrm{Gr}^0_{\mathrm{res}}$ *of the restricted Grassmannian is a homogeneous space under the unitary group* $\mathrm{U}_{1,2}\subset\mathrm{U}_2$.

Proof. The restricted Grassmannian is a symmetric space of the restricted unitary group $\mathrm{U}_{\mathrm{res}}$. It follows from the description of geodesics in Proposition 8.8 in [Ar03] (see also [ON83] and [CE75] or its infinite-dimensional version as given in Example 3.9 in [Ne02c], or Proposition 1.9 in [Tu06]) that each geodesic of $\mathrm{Gr}_{\mathrm{res}}$ starting at $W\in\mathrm{Gr}^0_{\mathrm{res}}$ is given by

$$\beta(t)=(\exp tX)\cdot\mathcal{H}_+,\quad X\in\mathfrak{m}_W, \tag{5.1}$$

where $\mathfrak{m}_W$ is the orthogonal in $\mathfrak{u}_{\mathrm{res}}$ to the Lie algebra of the isotropy group of W. For $W=\mathcal{H}_+$ we have $\mathfrak{m}=\mathfrak{u}(\mathcal{H})\cap(\mathfrak{S}_2(\mathcal{H}_+,\mathcal{H}_-)\oplus\mathfrak{S}_2(\mathcal{H}_-,\mathcal{H}_+))$, and for $W=g\cdot\mathcal{H}_+$ with $g\in\mathrm{U}_{\mathrm{res}}$, we have $\mathfrak{m}_W=g\,\mathfrak{m}\,g^{-1}$. Note that for $X\in\mathfrak{m}$, $\exp tX$ belongs to $\mathrm{U}_{1,2}\subset\mathrm{U}_2$. Since the Hopf-Rinow Theorem is no longer true in the infinite dimensional case, it is not clear whether every two elements in the complete connected manifold $\mathrm{Gr}^0_{\mathrm{res}}$ can be joined by a geodesic. Nevertheless Theorem B in [Ek78] asserts that, for every $W\in\mathrm{Gr}^0_{\mathrm{res}}$, the set of elements which can be joined to W by a unique minimal geodesic contains a dense G_δ set. Moreover from the properties of the Riemannian exponential map, there exists a neighborhood $\mathcal{V}$ of $\mathcal{H}_+$ in $\mathrm{Gr}^0_{\mathrm{res}}$ such that every element in $\mathcal{V}$ can be joined to $\mathcal{H}_+$ be a (minimal) geodesic. Hence an arbitrary element $W\in\mathrm{Gr}^0_{\mathrm{res}}$ can be joined to an element $W'\in\mathcal{V}$ by a geodesic

$$\beta_1(t)=(\exp tX_1)\cdot W',\quad X_1\in\mathfrak{m}_{W'},\quad t\in[0\,,1],$$

and W' can be joined to $\mathcal{H}_+$ by a geodesic

$$\beta_2(t)=(\exp tX_2)\cdot\mathcal{H}_+,\quad X_2\in\mathfrak{m},\quad t\in[0\,,1].$$

Consequently

$$W=\beta_1(1)=(\exp X_1)\cdot W'=(\exp X_1)(\exp X_2)\cdot\mathcal{H}_+.$$

But X_1 belongs to $\mathfrak{m}_{W'}=\exp(X_2)\mathfrak{m}\exp(-X_2)$, hence

$$W=(\exp X_2\exp X_3)\cdot\mathcal{H}_+$$

where $X_3=\mathrm{Ad}\,(\exp(-X_2))\,(X_1)$ belongs to $\mathfrak{m}$. Since $\exp X_3$ and $\exp X_2$ are elements of the unitary group $\mathrm{U}_{1,2}$, it follows that their product belongs to $\mathrm{U}_{1,2}$. Thus $\mathrm{U}_{1,2}$ acts transitively on $\mathrm{Gr}^0_{\mathrm{res}}$. □

Theorem 5.3. *The connected component* $\mathrm{Gr}^0_{\mathrm{res}}$ *of the restricted Grassmannian is a strong symplectic leaf in the Banach Lie-Poisson space* $\tilde{\mathfrak{u}}_2$. *More precisely, for every* $\gamma\neq 0$, *the* U_2*-affine coadjoint orbit* $\tilde{\mathcal{O}}_{(0,\gamma)}$ *of* $(0,\gamma)\in\tilde{\mathfrak{u}}_2$ *is diffeomorphic to* $\mathrm{Gr}^0_{\mathrm{res}}$ *via the application*

$$\begin{aligned}\Phi_\gamma \mathrm{Gr}^0_{\mathrm{res}} &\to \mathcal{O}_{(0,\gamma)}\\ W &\mapsto 2\mathrm{i}\gamma(p_W-p_+),\end{aligned}$$

where p_W *denotes the orthogonal projection on* W. *The pull-back by* Φ_γ *of the symplectic form on* $\tilde{\mathcal{O}}_{(0,\gamma)}$ *is* $(\gamma/2)$*-times the symplectic form* ω_{Gr} *on* $\mathrm{Gr}^0_{\mathrm{res}}$.

Proof. The assertion follows by the method of proof of Theorem 2.13, since $\mathrm{Gr}^0_{\mathrm{res}}$ is transitively acted upon by the group U_2 according to Proposition 5.2. □

Next we shall investigate the existence of invariant complex structures on certain covering spaces of the symplectic leaves of $\tilde{\mathfrak{u}}_2$ (Corollary 5.6 below). To this end we need two facts holding in a more general setting. In connection with the first of these statements, we note that invariant complex structures on certain homogeneous spaces related to derivations of L^*-algebras have been previously obtained by a different method in Theorem IV.5 in [Ne00].

Proposition 5.4. *Let $\mathfrak{X}$ be a real Hilbert Lie algebra with a scalar product denoted by $(\cdot \mid \cdot)$. Assume that there exists a connected Hilbert Lie group $\mathrm{U}_{\mathfrak{X}}$ whose Lie algebra is $\mathfrak{X}$; we write $\mathbf{L}(\mathrm{U}_{\mathfrak{X}}) = \mathfrak{X}$.*

Now let $D\colon \mathfrak{X} \to \mathfrak{X}$ be a bounded linear derivation such that

$$(\forall x, y \in \mathfrak{X}) \qquad (Dx \mid y) = -(x \mid Dy). \tag{5.2}$$

Consider the closed subalgebra $\mathfrak{h}_0 := \operatorname{Ker} D$ of $\mathfrak{X}$ and define

$$H_0 := \langle \exp_{\mathrm{U}_{\mathfrak{X}}}(\mathfrak{h}_0)\rangle,$$

that is, the subgroup of $\mathrm{U}_{\mathfrak{X}}$ generated by the image of $\mathfrak{h}_0$ by the exponential map.

If it happens that H_0 is a Lie subgroup of $\mathrm{U}_{\mathfrak{X}}$, then the smooth homogeneous space $\mathrm{U}_{\mathfrak{X}}/H_0$ has an invariant complex structure.

Proof. Denote $\mathfrak{L} := \mathfrak{X}_{\mathbb{C}}$, that is, the complex Hilbert-Lie algebra which is the complexification of $\mathfrak{X}$ and is endowed with the complex scalar product $(\cdot \mid \cdot)$ extending the scalar product of $\mathfrak{X}$. We denote the complex linear extension of D to $\mathfrak{L}$ again by D.

Then $D^* = -D$ as operators on the complex Hilbert space $\mathfrak{L}$, so that $-\mathrm{i}D \in \mathcal{B}(\mathfrak{L})$ is a self-adjoint operator. Let us denote its spectral measure by $\delta \mapsto E(\delta)$. Thus $E(\cdot)$ is a spectral measure on $\mathbb{R}$ and we have

$$D = \mathrm{i}\int_{\mathbb{R}} t\mathrm{d}E(t).$$

Also denote $S = (-\infty, 0]$, which is a closed subsemigroup of $\mathbb{R}$, and

$$\mathfrak{k} := \operatorname{Ran} E(-S) = \operatorname{Ran} E([0,\infty)) \subseteq \mathfrak{L}.$$

Then $\mathfrak{k}$ is a closed subspace of $\mathfrak{L}$ since it is the range of an idempotent continuous map. In addition, since D is a derivation of the Hilbert Lie algebra $\mathfrak{X}$ and S is a closed semigroup, it follows by Proposition 6.4 in [Be06] that $\mathfrak{k}$ is a complex subalgebra of $\mathfrak{L}$ with the following properties:

(i) $[\mathfrak{h}_0, \mathfrak{k}] \subseteq \mathfrak{k}$,
(ii) $\mathfrak{k} \cap \bar{\mathfrak{k}} = \mathfrak{h}_0 + \mathrm{i}\mathfrak{h}_0$ $(= \operatorname{Ker} D)$, and
(iii) $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{L}$.

Moreover, for every $y \in \mathfrak{h}_0$ and all $x \in \mathfrak{X}$ we have

$$D[y,x] = [Dy,x] + [y,Dx] = [y,Dx]$$

since $Dy = 0$. Therefore, we have $D \circ \mathrm{ad}_{\mathfrak{X}} y = \mathrm{ad}_{\mathfrak{X}} y \circ D$ for each $y \in \mathfrak{h}_0$. According to the definition of H_0, it then follows that for arbitrary $h \in H_0$ we have $\mathrm{Ad}_{\mathrm{U}_{\mathfrak{X}}} h \circ D = D \circ \mathrm{Ad}_{\mathrm{U}_{\mathfrak{X}}} h$ on $\mathfrak{X}$. Then the latter equality holds throughout $\mathfrak{L}$, and it then follows that the operator $\mathrm{Ad}_{\mathrm{U}_{\mathfrak{X}}} h\colon \mathfrak{L} \to \mathfrak{L}$ commutes with every value of the spectral measure $E(\cdot)$. In particular we have $\mathrm{Ad}_{\mathrm{U}_{\mathfrak{X}}}(h) \circ E(-S) = E(-S) \circ \mathrm{Ad}_{\mathrm{U}_{\mathfrak{X}}}(h)$, whence

(i') $(\forall h \in H_0) \qquad \mathrm{Ad}_{\mathrm{U}_{\mathfrak{X}}}(h)\mathfrak{k} \subseteq \mathfrak{k}$.

Now Theorem 6.1 in [Be06] shows that the smooth homogeneous space $\mathrm{U}_{\mathfrak{X}}/H$ has an invariant complex structure. □

Proposition 5.5. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $a \in \mathcal{B}(\mathcal{H})$ such that $a^* = -a$. Denote by*

$$D = \mathrm{ad}_{\mathfrak{u}_2} a\colon \mathfrak{u}_2 \to \mathfrak{u}_2, \quad x \mapsto [a,x]$$

the derivation of the compact L^*-algebra $\mathfrak{u}_2$ defined by a, and denote

$$\mathfrak{h}_0 := \operatorname{Ker} D = \{x \in \mathfrak{u}_2 \mid [a,x] = 0\}.$$

Next denote

$$H := \{u \in \mathrm{U}_2 \mid uau^{-1} = a\}$$

and in addition define

$$H_0 := \langle \exp(\mathfrak{h}_0)\rangle.$$

That is, H_0 is the subgroup of U_2 *generated by the image of* $\mathfrak{h}_0$ *by the exponential map. Then the following assertions hold:*

(j) *Both H and H_0 are Lie subgroups of* U_2.
(jj) *The subgroup H_0 is the connected component of* $\mathbf{1} \in H$.
(jjj) *The natural map*
$$\mathrm{U}_2/H_0 \to \mathrm{U}_2/H, \quad uH_0 \mapsto uH,$$
is an U_2*-equivariant smooth covering map.*

Proof. Consider the Banach algebra $\mathcal{A} := \mathbb{C}\mathbf{1} + \mathfrak{S}_2(\mathcal{H})$ and denote by $\varphi\colon \mathcal{A} \to \mathbb{C}$ the continuous linear functional uniquely defined by the conditions $\varphi(\mathbf{1}) = 1$ and $\operatorname{Ker}\varphi = \mathfrak{S}_2(\mathcal{H})$. Then we have
$$H = \{u \in \mathcal{A}^\times \mid u^*u = uu^* = \mathbf{1} \text{ and } \varphi(u) = 1\}$$
hence H is a Lie subgroup of $\mathcal{A}^\times$ by the Harris-Kaup theorem (see for instance Theorem 4.13 in [Be06]), and in addition the Lie algebra of H is
$$\mathbf{L}(H) = \{x \in \mathcal{A} \mid x^* = -x \text{ and } xa = ax\} = \mathfrak{h}_0.$$
On the other hand, H_0 has the structure of connected Lie group such that the inclusion map $H_0 \hookrightarrow \mathrm{U}_2$ is an immersion and $\mathbf{L}(H_0) = \mathfrak{h}_0$. (See for instance Theorem 3.5 in [Be06] and its proof.) Since $H_0 \subseteq H$ and $\mathbf{L}(H_0) = \mathbf{L}(H) = \mathfrak{h}_0$, it then follows that H_0 is the connected component of $\mathbf{1} \in H$. This can be seen directly by Lie theoretic methods; specifically, one just has to use the fact that the exponential map of any Banach Lie group is a local diffeomorphism at 0. An alternative approach is to use the proof of Lie's second theorem by means of the Frobenius theorem (see for instance Theorem 5.4 in Chapter VI of [La01]). According to that proof, the connected group H_0 is the integral manifold through $\mathbf{1}$ corresponding to a smooth left-invariant integrable distribution on U_2 whose fiber at $\mathbf{1}$ is (the complemented closed Lie subalgebra) $\mathfrak{h}_0$. Now recall the universality property of the integral leaves of integrable distributions according to Theorem 4.2 in Chapter VI of [La01] or, more generally, Theorem 4(iii) in [Nu92], which implies that the inclusion map $H_0 \hookrightarrow H$ is smooth. Then the wished-for property that H_0 is open in H follows since H_0 and H have the same tangent space at $\mathbf{1} \in H_0 \subseteq H$.

By either of these methods it follows that H_0 is an open subgroup of the Lie subgroup H of U_2, and then H_0 is in turn a Lie subgroup of U_2. Thus assertions (j) and (jj) are proved. Assertion (jjj) follows since the natural map $\mathrm{U}_2/H_0 \to \mathrm{U}_2/H$ is clearly an U_2-equivariant map whose tangent map at every point is an isomorphism. □

Corollary 5.6. *Every symplectic leaf of the Hilbert Lie-Poisson space $\tilde{\mathfrak{u}}_2$ is transitively acted on by* U_2 *by means of the affine coadjoint action and is* U_2*-equivariantly covered by some complex homogeneous space of* U_2.

Proof. Let $(\mu, \gamma) \in \tilde{\mathfrak{u}}_2$ arbitrary and denote $a := \mu - \gamma d \in \mathcal{B}(\mathcal{H})$. With the notation of Proposition 5.5, it is clear that H is equal to the isotropy group of the affine coadjoint action of U_2. Thus the symplectic leaf $\tilde{\mathcal{O}}_{(\mu,\gamma)}$ through (μ, γ) is U_2-equivariantly diffeomorphic to U_2/H. Now the conclusion follows since U_2/H is U_2-equivariantly covered by the complex homogeneous space U_2/H_0, according to Propositions 5.4 and 5.5. □

Remark 5.7. It follows by Corollary 5.6 that every simply connected symplectic leaf of the Banach Lie-Poisson space $\tilde{\mathfrak{u}}_2$ has an U_2-invariant complex structure. For instance, this is the case for the connected component $\mathrm{Gr}^0_{\mathrm{res}}$ of the restricted Grassmannian viewed as a symplectic leaf of $\tilde{\mathfrak{u}}_2$ by means of Theorem 5.3.

6. Some pathological properties of the restricted algebras

6.1. **Unbounded unitary groups in the restricted algebra.** We are going to point out a property that provides a good illustration for the difference between the Banach $*$-algebra $\mathcal{B}_{\mathrm{res}}$ and a C^*-algebra (Proposition 6.2 below).

Lemma 6.1. *Let $a \in \mathcal{B}(\mathcal{H}_-, \mathcal{H}_+)$ and assume that $a = v|a|$ and $a^* = w|a^*|$ are the polar decompositions of a and a^*, where $|a| \in \mathcal{B}(\mathcal{H}_-)$ and $|a^*| \in \mathcal{B}(\mathcal{H}_+)$, while $v\colon \mathcal{H}_- \to \mathcal{H}_+$ and $w\colon \mathcal{H}_+ \to \mathcal{H}_-$ are partial isometries. Next, denote*
$$\rho = \begin{pmatrix} 0 & a \\ -a^* & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}).$$

Then

$$\exp\rho = \begin{pmatrix} \cos|a^*| & v\sin|a| \\ -w\sin|a^*| & \cos|a| \end{pmatrix}.$$

Proof. We have

$$\rho^2 = \begin{pmatrix} -aa^* & 0 \\ 0 & -a^*a \end{pmatrix} = -\begin{pmatrix} |a^*|^2 & 0 \\ 0 & |a|^2 \end{pmatrix}$$

hence

$$(\forall n\geq 0)\quad \rho^{2n} = (-1)^n \begin{pmatrix} |a^*|^{2n} & 0 \\ 0 & |a|^{2n} \end{pmatrix}.$$

This implies that for every $n\geq 0$ we have

$$\rho^{2n+1} = \rho\cdot\rho^{2n} = (-1)^n \begin{pmatrix} 0 & v|a| \\ -w|a^*| & 0 \end{pmatrix} \begin{pmatrix} |a^*|^{2n} & 0 \\ 0 & |a|^{2n} \end{pmatrix} = (-1)^n \begin{pmatrix} 0 & v|a|^{2n+1} \\ -w|a^*|^{2n+1} & 0 \end{pmatrix}.$$

Consequently

$$\exp\rho = \sum_{n=0}^{\infty}\Bigl(\frac{1}{(2n)!}\rho^{2n} + \frac{1}{(2n+1)!}\rho^{2n+1}\Bigr) = \begin{pmatrix} \cos|a^*| & v\sin|a| \\ -w\sin|a^*| & \cos|a| \end{pmatrix}$$

which concludes the proof. □

Proposition 6.2. *All of the unitary groups* $(\mathbf{1}+\mathcal{F})\cap \mathrm{U}(\mathcal{H})$, $\mathrm{U}_{1,2}$, *and* $\mathrm{U}_{\mathrm{res}}$ *are unbounded subsets of the unital associative Banach algebra* $\mathcal{B}_{\mathrm{res}}$.

Proof. We have

$$(\mathbf{1}+\mathcal{F})\cap \mathrm{U}(\mathcal{H}) \subseteq \mathrm{U}_{1,2} \subseteq \mathrm{U}_{\mathrm{res}}$$

so it suffices to show that

$$\sup\{\|u\|_{\mathrm{res}} \mid u\in(\mathbf{1}+\mathcal{F})\cap\mathrm{U}(\mathcal{H})\} = \infty. \tag{6.1}$$

To this end let $n\geq 1$ be an arbitrary positive integer, pick a projection $q_n = q_n^* = q_n^2 \in \mathcal{B}(\mathcal{H}_-)$ with $\dim(\mathrm{Ran}\, q_n) = n$ and define $a_n := v_n((\pi/2)q_n) = (\pi/2)v_n \in \mathcal{B}(\mathcal{H}_-,\mathcal{H}_+)$, where $v_n\colon \mathcal{H}_-\to\mathcal{H}_+$ is an arbitrary partial isometry such that $v_n^*v_n = q_n$. Then $|a_n| = (\pi/2)q_n$, so that $\sin|a_n| = q_n$ and then $\|(\sin|a_n|)\|_2 = \sqrt{\dim(\mathrm{Ran}\, q_n)} = \sqrt{n}$. Now Lemma 6.1 shows that the element

$$\rho_n = \begin{pmatrix} 0 & a_n \\ -a_n^* & 0 \end{pmatrix} \in \mathfrak{u}(\mathcal{H})\cap\mathcal{F}$$

satisfies

$$\|\exp(\rho_n)\|_{\mathrm{res}} \geq \|(\sin|a_n|)\|_2 = \sqrt{n}.$$

Now the desired conclusion (6.1) follows since $\exp(\rho_n)\in(\mathbf{1}+\mathcal{F})\cap\mathrm{U}(\mathcal{H})$ and $n\geq 1$ is arbitrary. □

6.2. **The predual of the restricted algebra is not spanned by its positive cone.** It is well known that every self-adjoint normal functional in the predual of a W^*-algebra can be written as the difference of two positive normal functionals. It is also well known and easy to see that a similar property holds for the preduals of numerous operator ideals. More precisely, if $\mathfrak{J}$ and $\mathfrak{B}$ are Banach operator ideals such that the trace pairing

$$(\mathfrak{B},\mathfrak{J})\to\mathbb{C},\quad (T,S)\mapsto \mathrm{Tr}\,(TS)$$

is well defined and induces a topological isomorphism of the topological dual $\mathfrak{B}^*$ onto $\mathfrak{J}$, then for every $T=T^*\in\mathfrak{B}$ there exist $T_1,T_2\in\mathfrak{B}$ such that $T_1\geq 0$, $T_2\geq 0$ and $T=T_1-T_2$. In fact, we can take $T_1 = (|T|+T)/2$ and $T_2 = (|T|-T)/2$, and we have $T_1,T_2\in\mathfrak{B}$ since $|T|\in\mathfrak{B}$. (The latter property follows since if $T=W|T|$ is the polar decomposition of T, then $|T| = W^*T\in\mathfrak{B}$.)

We shall see in Proposition 6.4 below that the predual $(\mathfrak{u}_{\mathrm{res}})_*$ of the restricted Lie algebra fails to have the similar property of being spanned by its elements ρ with $\mathrm{i}\rho\geq 0$. In fact, the linear span of these elements turns out to be the proper subspace $\mathfrak{u}_1$ of $(\mathfrak{u}_{\mathrm{res}})_*$.

Lemma 6.3. *Let* $\mathcal{H}_\pm$ *be two complex separable Hilbert spaces,* $\mathcal{H} = \mathcal{H}_+\oplus\mathcal{H}_-$, $0\leq a_\pm\in\mathcal{B}(\mathcal{H}_\pm)$, *and* $t\in\mathcal{B}(\mathcal{H}_-,\mathcal{H}_+)$. *Also denote*

$$a = \begin{pmatrix} a_+ & t \\ t^* & a_- \end{pmatrix} \in \mathcal{B}(\mathcal{H}).$$

Then the following assertions hold:

(i) *We have $a \geq 0$ if and only if the inequality*

$$|\langle \xi, t\eta\rangle|^2 \leq \langle \xi, a_+\xi\rangle \cdot \langle \eta, a_-\eta\rangle \tag{6.2}$$

holds for all $\xi \in \mathcal{H}_+$ and $\eta \in \mathcal{H}_-$.

(ii) *If $a \geq 0$ and in addition $a_\pm \in \mathfrak{S}_1(\mathcal{H}_\pm)$ and $t \in \mathfrak{S}_2(\mathcal{H}_-, \mathcal{H}_+)$, then*

$$\|t\|_2 \leq (\mathrm{Tr}\, a)/\sqrt{2}. \tag{6.3}$$

Proof. For assertion (i) see Exercise 3.2 at the end of Chapter 3 in [Pa02].

Next, let $\{\xi_i\}_{i\geq 1}$ and $\{\eta_j\}_{j\geq 1}$ be orthonormal bases in the Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. Then (6.2) shows that

$$(\forall i, j \geq 1) \quad |\langle \xi_i, t\eta_j\rangle|^2 \leq \langle \xi_i, a_+\xi_i\rangle \cdot \langle \eta_j, a_-\eta_j\rangle.$$

Now recall that $(\|t\|_2)^2 = \sum\limits_{i,j\geq 1} |\langle \xi_i, t\eta_j\rangle|^2$, $\mathrm{Tr}\, a_+ = \sum\limits_{i\geq 1} \langle \xi_i, a_+\xi_i\rangle$, and $\mathrm{Tr}\, a_- = \sum\limits_{j\geq 1} \langle \eta_j, a_-\eta_j\rangle$. Thus, adding the above inequalities, we get

$$(\|t\|_2)^2 \leq (\mathrm{Tr}\, a_+) \cdot (\mathrm{Tr}\, a_-) \leq (\mathrm{Tr}\, a_+ + \mathrm{Tr}\, a_-)^2/2 = (\mathrm{Tr}\, a)^2/2$$

and assertion (ii) follows. ☐

Proposition 6.4. *The following assertions hold:*

(i) *If $a \in (\mathfrak{u}_{\mathrm{res}})_*$ and $\mathrm{i}a \geq 0$, then $a \in \mathfrak{S}_1(\mathcal{H})$ and $\|a\|_1 \leq \|a\|_{(\mathfrak{u}_{\mathrm{res}})_*} \leq (1+\sqrt{2})\|a\|_1$.*

(ii) *If $\rho \in (\mathfrak{u}_{\mathrm{res}})_* \setminus \mathfrak{u}_1$ then there exist no $\rho_1, \rho_2 \in (\mathfrak{u}_{\mathrm{res}})_*$ such that $\mathrm{i}\rho_1 \geq 0$, $\mathrm{i}\rho_2 \geq 0$, and $\rho = \rho_1 - \rho_2$.*

Proof. (i) Let $a \in (\mathfrak{u}_{\mathrm{res}})_*$ such that $\mathrm{i}a \geq 0$, and denote $\mathrm{i}a =: \begin{pmatrix} a_+ & t \\ t^* & a_- \end{pmatrix}$. Then

$$\begin{aligned} \|a\|_1 &= \|\mathrm{i}a\|_1 = \mathrm{Tr}\,(\mathrm{i}a) = \mathrm{Tr}\, a_+ + \mathrm{Tr}\, a_- = \|a_+\|_1 + \|a_-\|_1 \\ &\leq \|\mathrm{i}a\|_{(\mathfrak{u}_{\mathrm{res}})_*} = \|a_+\|_1 + \|a_-\|_1 + 2\|t\|_2 \\ &\leq \|a_+\|_1 + \|a_-\|_1 + \sqrt{2}\cdot \mathrm{Tr}\,(\mathrm{i}a) = (1+\sqrt{2})\|\mathrm{i}a\|_1 = (1+\sqrt{2})\|a\|_1, \end{aligned}$$

where the second inequality follows by Lemma 6.2(ii). Consequently, for all $a \in (\mathfrak{u}_{\mathrm{res}})_*$ with $\mathrm{i}a \geq 0$ we have $\|a\|_1 \leq \|a\|_{(\mathfrak{u}_{\mathrm{res}})_*} \leq (1+\sqrt{2})\|a\|_1$.

(ii) Let $\rho \in (\mathfrak{u}_{\mathrm{res}})_* \setminus \mathfrak{u}_1$ and assume that there exist elements $\rho_1, \rho_2 \in (\mathfrak{u}_{\mathrm{res}})_*$ such that $\mathrm{i}\rho_1 \geq 0$, $\mathrm{i}\rho_2 \geq 0$, and $\rho = \rho_1 - \rho_2$. Then $\mathrm{i}\rho_1, \mathrm{i}\rho_2 \in \mathfrak{S}_1(\mathcal{H})$ according to the assertion (i), which we have already proved. Consequently, $\rho_1, \rho_2 \in \mathfrak{u}_1$, whence $\rho = \rho_1 - \rho_2 \in \mathfrak{u}_1$. This is a contradiction with the assumption on ρ, which concludes the proof. ☐

6.3. **The Cartan subalgebras of $\mathfrak{u}_{\mathrm{res}}$ are not $\mathrm{U}_{\mathrm{res}}$-conjugate.** For a (finite-dimensional) compact connected semi-simple Lie subgroup G of the unitary group $\mathrm{U}(n)$, every element X of the Lie algebra $\mathfrak{g}$ of G is conjugate to a diagonal element by an element of G. This can be seen as follows (see [He62] chap. V theorem 6.4 for more general results). Take a diagonal element $H \in \mathfrak{g}$ such that the one-parameter subgroup $\exp tH$ is dense in the torus whose Lie algebra is the set of diagonal matrices belonging to $\mathfrak{g}$. On G, consider the continuous function $g \mapsto \mathrm{B}\,(H, \mathrm{Ad}(g)(X))$, where B denotes the Killing form of G. By compactness, this function takes a minimum at some g_0, and for every element Y in $\mathfrak{g}$ one has

$$\frac{d}{dt}\mathrm{B}\,(H, \mathrm{Ad}(\exp tY)\mathrm{Ad}(g_0)(X))|_{t=0} = 0,$$

i.e $\mathrm{B}\,(H, [Y, \mathrm{Ad}(g_0)(X)]) = 0$. Since the Killing form is $\mathrm{Ad}(G)$-invariant, one has

$$\mathrm{B}\,(H, [Y, \mathrm{Ad}(g_0)(X)]) = \mathrm{B}\,([\mathrm{Ad}(g_0)(X), H], Y)\,.$$

The non-degeneracy of the Killing form then implies that $[\mathrm{Ad}(g_0)(X), H] = 0$. But H has been chosen such that the centralizer of H is the set of diagonal matrices belonging to $\mathfrak{g}$. Consequently $\mathrm{Ad}(g_0)(X)$ is a diagonal element in $\mathfrak{g}$. It follows that the maximal Abelian subalgebras, called *Cartan subalgebras*, of $\mathfrak{g}$ are conjugate under G. Naturally this proof does not work anymore for an infinite-dimensional group since the argument to minimize the corresponding function is missing. In fact, we will show below that the Cartan subalgebras of $\mathfrak{u}_{\mathrm{res}}$ are not $\mathrm{U}_{\mathrm{res}}$-conjugate, in general.

We note that a related fact follows from results in the paper [BS96]. Specifically, let $\rho_0 \in (\mathfrak{u}_{\rm res})_*$ such that $[d, \rho_0] = 0$, $\operatorname{Ker}\rho_0 = \{0\}$, and each eigenvalue of ρ_0 has multiplicity 1. Next denote by $\mathcal{O}_{\rho_0}$ the coadjoint $\mathrm{U}_{\rm res}$-orbit of ρ_0, let $\rho \in (\mathfrak{u}_{\rm res})_*$, and define

$$f_\rho \colon \mathcal{O}_{\rho_0} \to (0, \infty), \quad f_\rho(b) = \|\rho - b\|_2.$$

If the function f_ρ happens to have a critical point $\rho_1 \in \mathcal{O}_{\rho_0}$, then $[\rho_1, \rho] = 0$ according to [BS96]. Since $\rho_1 \in \mathcal{O}_{\rho_0}$, there exists $u \in \mathrm{U}_{\rm res}$ such that $\rho_1 = u\rho_0 u^{-1}$, and then $[\rho_0, u^{-1}\rho u] = 0$. The latter equality implies that $u^{-1}\rho u$ commutes with all of the spectral projections of ρ_0. Hence $[d, u^{-1}\rho u] = 0$ in view of the spectral assumptions on ρ_0, and then Proposition 3.3 applied to $u^{-1}\rho u$ shows that the coadjoint isotropy group of ρ is a Banach-Lie subgroup of $\mathrm{U}_{\rm res}$ and the corresponding $\mathrm{U}_{\rm res}$-coadjoint orbit $\mathcal{O}_\rho$ is a smooth leaf of the characteristic distribution of $(\mathfrak{u}_{\rm res})_*$.

Proposition 6.5. *The unitary group $\mathrm{U}_{\rm res}$ does not act transitively on the set of Cartan subalgebras of its Lie algebra.*

Proof. Endow the Hilbert space $\mathcal{H}$ with an orthonormal basis $\mathcal{B} = \{e_n \mid n \in \mathbb{Z}^*\}$, such that $\{e_n \mid n \in -\mathbb{N}^*\}$ is an orthonormal basis of $\mathcal{H}_+$ and $\{e_n \mid n \in \mathbb{N}^*\}$ an orthonormal basis of $\mathcal{H}_-$. The set $\mathcal{D}$ of skew-Hermitian bounded diagonal operators with respect to $\mathcal{B}$ form a Cartan subalgebra of $\mathfrak{u}_{\rm res}$. Now consider the following subset of the set of anti-diagonal elements in $\mathfrak{u}_{\rm res}$:

$$\mathcal{J} = \{J \in \mathfrak{u}_{\rm res} \mid J(e_n) \in \mathbb{R}e_{-n} \ \forall n \in \mathbb{Z}^*\}.$$

Since the coefficients $J_{-k,k}$, $k \in \mathbb{Z}^*$, of $J \in \mathcal{J}$ satisfy $J_{-k,k} = -J_{k,-k}$, it follows from an easy computation that $\mathcal{J}$ is Abelian. An element $B = (B_{i,j}) \in \mathfrak{u}_{\rm res}$ commutes with every element $J = (J_{i,j})$ in $\mathcal{J}$ if and only if

$$([B, J]_{i,-k}) = (B_{i,k}J_{k,-k} - J_{i,-i}B_{-i,-k}) \tag{6.4}$$

vanishes for every $J \in \mathcal{J}$. This implies the following conditions:

$$\begin{aligned} B_{i,k} &= 0 \ \text{ for } \ i \notin \{k, -k\}; \\ B_{k,k} &= B_{-k,-k} \ \text{ for } k \in \mathbb{Z}^*; \\ B_{-k,k} &= -B_{k,-k} \ \text{ for } k \in \mathbb{Z}^*. \end{aligned}$$

It follows that the maximal Abelian subalgebra $\mathcal{C}$ of $\mathfrak{u}_{\rm res}$ which contains $\mathcal{J}$ is $\mathcal{J} + \mathcal{D}_+$, where

$$\mathcal{D}_+ = \{D = (D_{i,j}) \in \mathcal{D} \mid D_{-k,-k} = D_{k,k} \ \forall k \in \mathbb{Z}^*\}.$$

Let us prove by contradiction that the Cartan subalgebras $\mathcal{C}$ and $\mathcal{D}$ are not conjugate under $\mathrm{U}_{\rm res}$. Suppose that there exists a unitary operator

$$g = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix} \in \mathrm{U}_{\rm res}$$

such that $g\mathcal{J}g^{-1} = \mathcal{D}$. Consider an element

$$J = \begin{pmatrix} 0 & J_{+-} \\ J_{-+} & 0 \end{pmatrix} \in \mathcal{J}$$

which is a Hilbert-Schmidt operator that is not trace class. One has

$$\begin{aligned} gJg^{-1} &= \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix} \begin{pmatrix} 0 & J_{+-} \\ J_{-+} & 0 \end{pmatrix} \begin{pmatrix} g_{++}^* & g_{-+}^* \\ g_{+-}^* & g_{--}^* \end{pmatrix} \\ &= \begin{pmatrix} g_{+-}J_{-+}g_{++}^* + g_{++}J_{+-}g_{+-}^* & g_{+-}J_{-+}g_{-+}^* + g_{++}J_{+-}g_{--}^* \\ g_{--}J_{-+}g_{++}^* + g_{-+}J_{+-}g_{+-}^* & g_{--}J_{-+}g_{-+}^* + g_{-+}J_{+-}g_{--}^* \end{pmatrix}. \end{aligned}$$

By hypothesis, gJg^{-1} is a diagonal operator

$$D = \begin{pmatrix} D_{++} & 0 \\ 0 & D_{--} \end{pmatrix}$$

with $D_{++} = g_{+-}J_{-+}g_{++}^* + g_{++}J_{+-}g_{+-}^*$ and $D_{--} = g_{--}J_{-+}g_{-+}^* + g_{-+}J_{+-}g_{--}^*$. Now, since g belongs to $\mathrm{U}_{\rm res}$, g_{+-} and g_{-+} are Hilbert-Schmidt. Since J belongs to $\mathfrak{S}_2(\mathcal{H})$, J_{+-} and J_{-+} are Hilbert-Schmidt as well. From the relation $\mathfrak{S}_2 \cdot \mathfrak{S}_2 \subset \mathfrak{S}_1$, it follows that D_{++} and D_{--} are trace class, hence D belongs to $\mathfrak{S}_1(\mathcal{H})$. But this implies that $J = g^{-1}Dg$ is also trace class, since $\mathfrak{S}_1(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$. This leads to a contradiction by the choice of $J \in \mathcal{J}$. It follows that elements in $\mathcal{J} \setminus \mathfrak{S}_1(\mathcal{H})$ are not $\mathrm{U}_{\mathrm{res}}$-conjugate to diagonal elements. Consequently, the Cartan subalgebra $\mathcal{C}$ and $\mathcal{D}$ are not $\mathrm{U}_{\mathrm{res}}$-conjugate. □

Remark 6.6. Since every skew-Hermitian operator is conjugate to a diagonal operator by a unitary operator, the set of conjugacy classes of Cartan subalgebras in $\mathfrak{u}_{\mathrm{res}}$ is in bijection with $\mathrm{U}(\mathcal{H})/\mathrm{U}_{\mathrm{res}}$ and is infinite. The conjugacy classes of Cartan subalgebras are related to the conjugacy classes of maximal tori. An infinite number of conjugacy classes of maximal tori has already been encountered in the case of some groups of contactomorphisms (see [Le01]). Examples of maximal tori of different dimensions were provided in [HT03] in some groups of symplectomorphisms.

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Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail address: Daniel.Beltita@imar.ro

Section de Mathématiques and Centre Bernoulli, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

E-mail address: Tudor.Ratiu@epfl.ch

Section de Mathématiques, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

E-mail address: Alice.Tumpach@epfl.ch