Root theory of L^*-algebras and applications

A. B. Tumpach

The topic of our research is to do (some) geometry on infinite-dimensional manifolds like the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$, or the restricted Siegel disc $\mathrm{D}_{\mathrm{res}}$. The restricted Grassmannian is related to loop groups ([10], [14]), hierarchies of equations of KdV-type ([13]), and Fermionic second quantization ([21]). The restricted Siegel disc contains in a natural way the universal Teichmüller space $T_0(1)$ (with the Hilbert manifold structure constructed in [16]) as well as the Teichmüller space of any compact Riemann surface.

In recent work ([17]) we construct a hyperkähler metric on the cotangent space of any infinite-dimensional Hermitian-symmetric affine coadjoint orbit of an L^*-group of compact type. An example of such an orbit is the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$. This metric is natural in the following sense: it is invariant under the L^*-group under consideration, restricts to the Kähler metric of the orbit and is compatible with the complex symplectic form of the cotangent space. In the case of the restricted Grassmannian, we obtain a hyperkähler metric on the cotangent space $T^*\mathrm{Gr}_{\mathrm{res}}$, which is invariant under the group U_2 of unitary operators which differ from the identity by Hilbert-Schmidt operators, restricts to the natural Kähler metric of the restricted Grassmannian and is compatible with the complex-symplectic structure of the cotangent space. This result is a generalization of finite-dimensional results in [4], [5] and [6] to the infinite-dimensional setting where the root theory of L^*-algebras is extensively used. The proof goes by identifying the cotangent space of a Hermitian-symmetric affine coadjoint orbit of an L^*-group of compact type with the orbit of the complexified L^*-group, where a potential can be computed. For orbits of non-compact type (also known as symmetric Hilbert domains) this identification does not hold anymore and it is still work in progress (with T. Ratiu and F. Gay-Balmaz) to construct a natural hyperkähler metric on the cotangent space of any symmetric Hilbert domain such as the restricted Siegel disc $\mathrm{D}_{\mathrm{res}}$ for instance.

The background for this research is the root theory of simple separable L^*-algebras developed by Schue in 1960-1961. The objects of study are infinite-dimensional Hermitian-symmetric affine coadjoint orbits of L^*-groups, which one may want to classify. The classification of the irreducible orbits can be made as in the finite-dimensional case ([20]) using the notion of roots of non-compact type ([18]). Finite-dimensional examples include the 2-sphere and the hyperbolic space. The tool for the construction of the hyperkähler structures mentioned above is the existence of maximal sets of strongly orthogonal roots on one hand, and the existence of Cartan subalgebras adapted to a given Cartan decomposition on the other ([8]). Looking back at the 50 year old theory of L^*-algebras, some questions arise: given that even for simple L^*-algebras some automorphisms are not inner, what are the conjugacy classes of Cartan subalgebras by inner automorphisms? Is any Cartan subalgebra of a simple L^*-algebra the centralizer of one of its elements? The conjugacy classes of Cartan subalgebras of simple L^*-algebras under the whole group of automorphisms are characterized for complex L^*-algebras in [2] and for real simple L^*-algebras in [1] (see also [9] and [3] for similar questions on other infinite-dimensional algebras).

References

[1] H. R. Alagia, *Cartan subalgebras of Banach-Lie algebras of operators*, Pacific Journal of Math., Vol. 98, No. 1, (1982), 1–15.

[2] V. K. Balachandran, *Simple L^*-algebras of classical type*, Math. Ann. 180, (1969), 205–219.

[3] D. Beltiţă, T. Ratiu, A.B. Tumpach, *The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits*, J. Funct. Anal. 247 (2007), 138–168.

[4] O. Biquard, P. Gauduchon, *La métrique hyperkählérienne des orbites coadjointes de type symétrique d'un groupe de Lie complexe semi-simple*, C. R. Acad. Sci. Paris, t. 323, série I (1996), 1259–1264.

[5] O. Biquard, P. Gauduchon, *Hyperkähler metrics on cotangent bundles of Hermitian Symmetric spaces*, Geometry and Physics, *Lect. notes Pure Appl. Math.* Serie 184, Marcel Dekker (1996), 287–298.

[6] O. Biquard, P. Gauduchon, *Géométrie hyperkählérienne des espaces hermitiens symétriques complexifiés*, Séminaire de théorie spectrale et géométrie, Grenoble, Vol 16 (1998), 127–173.

[7] M. Gordina, *Hilbert-Schmidt groups as infinite-dimensional Lie groups and their Riemannian geometry*, J. Funct. Anal. 227 (2005), no. 2, 245–272.

[8] K.-H. Neeb, *Highest weight representations and infinite-dimensional Kähler manifolds*, Recent advances in Lie theory (Vigo, 2000), 367–392, Res. Exp. Math., 25, Heldermann, Lemgo, 2002.

[9] K.-H. Neeb, I. Penkov, *Cartan subalgebras of $\mathfrak{gl}_\infty$*, Canad. Math. Bull. 46 (2003), no. 4, 597–616.

[10] A. Pressley, G. Segal, *Loop Groups*, Oxford Mathematical Monographs. Oxford (UK): Clarendon Press. viii, 318 p. (**1988**)

[11] J. R. Schue, *Hilbert space methods in the theory of Lie algebras*, Trans. Amer. Math. Soc. 95 (1960), 69–80.

[12] J. R. Schue, *Cartan decompositions for L^*-algebras*, Trans. Amer. Math. Soc 98, (1961), 334–349.

[13] G. Segal, G Wilson, *Loop Groups and equations of KdV type*, Terng, Chuu Lian (ed.) et al., Surveys in differential geometry, Vol. IV. A supplement to the Journal of Differential Geometry. Integral systems (integrable systems). Lectures on geometry and topology. Cambridge, MA: International Press, (**1998**), 403-466.

[14] M. Spera, T. Wurzbacher, *Differential geometry of Grassmannian embeddings of based loop groups*, Differential Geometry and its Applications **13**, (**2000**), 43-75, North-Holland.

[15] N. Stumme, *Automorphisms and conjugacy of compact real forms of the classical infinite dimensional matrix Lie algebras*, Forum Math. 13 (2001), no. 6, 817–851.

[16] L. A.Takhtajan, L. P. Teo, *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc. 183 (2006), no. 861, viii+119pp.

[17] A. B. Tumpach, *Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits*, Annales de l'Institut Fourier, Tome 59, (2009), fascicule 1, 167–197.

[18] A. B. Tumpach, *On the classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits*, Forum Math. 21 (2009), n0. 3, 375–393.

[19] J. A. Wolf, *Fine structure of Hermitian Symmetric Spaces*, Symmetric Spaces, short Courses presented at Washington Univ., pure appl. Math. 8, (1972), 271–357.

[20] J. A. Wolf, *On the classification of Hermitian Symmetric Spaces*, J. Math. Mech. 13, (1964), 489–496.

[21] T. Wurzbacher, *Fermionic Second Quantization and the Geometry of the Restricted Grassmannian*, in Infinite-Dimensional Kähler Manifolds, DMV Seminar, Band **31**, Birkhäuser, (**2001**).