Hyperkähler structures and infinite-dimensional Grassmannians

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Abstract

In this paper, we describe an example of a hyperkähler quotient of a Banach manifold by a Banach Lie group. Although the initial manifold is not diffeomorphic to a Hilbert manifold (not even to a manifold modelled on a reflexive Banach space), the quotient space obtained is a Hilbert manifold, which can be furthermore identified either with the cotangent space of a connected component Gr_{res}^j, ($j \in \mathbb{Z}$), of the restricted Grassmannian or with a natural complexification of this connected component, thus proving that these two manifolds are isomorphic hyperkähler manifolds. Moreover, Kähler potentials associated with the natural complex structure of the cotangent space of Gr_{res}^j and with the natural complex structure of the complexification of Gr_{res}^j are computed using Kostant-Souriau's theory of prequantization.

Résumé

Dans cet article, nous présentons un exemple de quotient hyperkählérien d'une variété banachique par un groupe de Lie banachique. Bien que la variété initiale ne soit pas difféomorphe à une variété hilbertienne (ni même à une variété modelée sur un espace de Banach réflexif), l'espace quotient obtenu est une variété hilbertienne, qui peut être identifiée, selon la structure complexe distinguée, soit à l'espace cotangent d'une composante connexe Gr_{res}^j ($j \in \mathbb{Z}$) de la grassmannienne restreinte, soit à une complexification naturelle de cette même composante connexe. De plus, les potentiels kählériens associés respectivement à la structure complexe naturelle de l'espace cotangent de Gr_{res}^j et à la structure complexe naturelle de la complexification de Gr_{res}^j sont calculés à l'aide de la théorie de préquantisation de Kostant-Souriau.

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Contents

1 Introduction

A Kähler manifold of finite dimension is a Riemannian manifold endowed with a complex structure that is parallel for the Levi-Civita connection, or equivalently, a manifold equipped with a closed symplectic real form, called the Kähler form, and a compatible integrable complex structure. A hyperkähler manifold of finite dimension is a manifold endowed with a Riemannian metric g and three complex structures I, J, K such that : $IJK = -1$, and g is Kähler with respect to each complex structure. Hence a hyperkähler manifold admits three Kähler forms ω_1, ω_2, et ω_3, and the choice of one complex structure, for instance I, allows to build a holomorphic symplectic form, namely $\Omega = \omega_2 + i\omega_3$.

D. Kaledin and B. Feix have found independently in [16] and [12] that, given a finite-dimensional manifold N endowed with a Kähler metric g_N, there exists a hyperkähler metric g defined on a neighborhood of the zero section of the cotangent space T^*N of N, compatible with the natural holomorphic symplectic structure of the cotangent space, and such that the restriction of g to N is g_N. In addition, g is unique if one requires S^1-invariance. D. Kaledin uses for his proof the theory of Hodge manifolds, whereas B. Feix uses twistor spaces. As far as we know, this result has not been extended to the infinite-dimensional Banach setting. Moreover this existence result does not lead to an explicit expression of the metric and examples of explicit hyperkähler metrics are rare.

O. Biquard and P. Gauduchon provided in [4] a construction of hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces, and in [5] hyperkähler metrics on coadjoint orbits of symmetric type of a complex semi-simple Lie group. Furthermore they established formulas for Kähler potentials that allow explicit expressions of these metrics. In [6] the same authors identified these hyperkähler structures, showing that the cotangent space and the complexified coadjoint orbit are, in the case of Hermitian symmetric spaces of finite dimension, two aspects of the same hyperkähler object, which appear in accordance with the chosen complex structure within the 2-sphere of complex structures.

In the case of the cotangent space of the Grassmannian $Gr(p,n)$ of subspaces of dimension p in $\mathbb{C}^n$, the aforementioned hyperkähler structure can be obtained by a hyperkähler quotient, and the corresponding Kähler potential can be computed via the theory of Kostant-Souriau's prequantization. It is the study of this particular example of Hermitian symmetric space that leads to the theory of O. Biquard and P. Gauduchon as developed in [4]. In the present work, we show that each connected component Gr_{res}^j ($j \in \mathbb{Z}$) of the restricted Grassmannian Gr_{res} introduced by Pressley and Segal in [38], gives an example of an infinite-dimensional Hermitian symmetric space whose cotangent space can be obtained by an infinite-dimensional hyperkähler quotient of a Banach manifold by a Banach Lie group. Moreover, we show that the resulting quotient space can also be identified with a natural complexification $\mathcal{O}_j^{\mathbb{C}}$ of Gr_{res}^j, also called complexified orbit, consisting of pairs (P, Q) of elements of Gr_{res}^j such that $P \cap Q^{\perp} = \{0\}$. In this way, the study of this particular example provides a first step towards the generalization of the aforementioned results of O. Biquard and P. Gauduchon to the infinite-dimensional setting. The full generalization of these results has been carried out in [49] by the construction of hyperkähler metrics on complexifications of Hermitian-symmetric affine coadjoint orbits of semi-simple L^*-groups of compact type, and by the identification of these complexifications with the cotangent spaces of the orbits under consideration. This generalization is based on Mostow's Decomposition Theorem (see [48]) and on the notion of strongly orthogonal roots of a L^*-algebra (see the Appendix in [49]).

The theory of symplectic quotients was initiated by J. E. Marsden and A. Weinstein in [28]. In finite dimension it was used in particular to construct new examples of symplectic manifolds. In [27] J. E. Marsden and T. Ratiu applied this theory to infinite-dimensional manifolds and obtained new developments of V. I. Arnold's idea that fluid motion equations are the equations of geodesics on a suitable infinite-dimensional Lie group. Another example of infinite-dimensional symplectic reduction is given by J. E. Marsden and A. Weinstein in [30] in relation with the Maxwell-Vlasov equation. (For an overview of applications of the symplectic reduction see [29].) Kähler and hyperkähler reductions are refinements of this theory. An infinite-dimensional version based on the study of Nahm's equations was used by P. B. Kronheimer in [22] and [23] in order to construct hyperkähler structures on maximal semi-simple and nilpotent coadjoint orbits of semi-simple complex (finite-dimensional) Lie groups. These results were generalized to all orbits by O. Biquard in [3] and A. G. Kovalev in [21].

In [15], V. G. Kac classifies infinite-dimensional Lie groups and algebras into four (overlapping) categories:
1) Groups of diffeomorphisms of manifolds and the corresponding Lie algebras of vector fields;
2) Lie groups (resp. Lie algebras) of maps from a finite-dimensional manifold to a finite-dimensional Lie group (resp. Lie algebra);
3) Classical Lie groups and algebras of operators on Hilbert and Banach spaces;
4) Kac-Moody algebras.

The examples of infinite-dimensional reduction mentioned above concern only the first two classes of groups. In this work we construct an example of hyperkähler reduction involving the third class of groups.

The structure of the paper is as follows. In Section 2, we introduce the necessary background on Kähler and hyperkähler quotients of Banach manifolds as well as the theory of Kähler potential on a quotient induced by Kostant-Souriau's theory of prequantization. In the infinite-dimensional setting, the definition of a hyperkähler manifold has to be specified so as to avoid problems such as the possible non-existence of a Levi-Civita connection for weak Riemannian metrics. The conditions needed to get a smooth Kähler structure on a Kähler quotient of a Banach manifold by a Banach Lie group have to be strengthened in comparison to the finite-dimensional case. In this Section, a basic definition of the notion of stable manifold associated with a level set and an holomorphic action of a complex Banach Lie group on a Kähler manifold is given. It is the most adapted to our purpose, but can be related to more sophisticated definitions as the one appearing in the Hilbert-Mumford Geometric Invariant Theory (see [33]) or the one appearing in Donaldson's work (see [11] for an exposition of the circle of ideas around this notion). In Subsection 2.2, the existence of a *smooth* projection of the stable manifold to the level set is proved and used in the identification of a smooth Kähler quotient with the complex quotient of the associated stable manifold by the complexification of the group. In Subsection 2.4, we give a survey of the theory of Kähler potential on a Kähler quotient which includes a natural generalization to the Banach setting of the formula for the Kähler potential proved by O. Biquard and P. Gauduchon in Theorem 3.1 in [4].

In Section 3, we construct a smooth hyperkähler quotient of the tangent bundle $T\mathcal{M}_k$ of a flat non-reflexive Banach space $\mathcal{M}_k$ (indexed by $k \in \mathbb{R}^*$) by a Banach Lie group G. The key point in the proof of this result is the existence of a G-equivariant slice of the tangent space to the level set, which is orthogonal to the G-orbits and allows one to define a structure of smooth Riemannian manifold on the quotient. As far as we know, the general procedure for finding closed complements to closed subspaces of a Banach space recently developed by D. Beltiţă and B. Prunaru in [2] does not apply in our cases, so that the existence of closed complements has to be worked out by hand. For this purpose, the properties of Schatten ideals are extensively used.

In Section 4, we show that the quotient space obtained in Section 3 can be identified with the cotangent bundle of a connected component Gr_{res}^j of the restricted Grassmannian , which is therefore endowed with a (1-parameter family of strong) hyperkähler structure(s). To prove this identification, we use the stable manifold associated with one of the complex structures of the quotient space and the general results of Section 2. At the end of Section 4, we compute the Kähler potential associated with this complex structure using the theory explained in Subsection 2.4, and we give an expression of this potential using the curvature of Gr_{res}. The formulas obtained by this method are analogous to the ones proved by O. Biquard and P. Gauduchon in the finite-dimensional setting. By restriction to the zero section, the theorems proved in Section 4 realize each connected component of the restricted Grassmannian as a Kähler quotient and provide the expression of the Kähler potential of the restricted Grassmannian induced by Plücker's embedding. The realization of the restricted Grassmannian as a symplectic quotient was independently obtained by T. Wurzbacher in unplublished work and explained in numerous talks (see [51]).

In Section 5, we show that the hyperkähler quotient constructed in Section 3 can also be identified with a natural complexification $\mathcal{O}_j^{\mathbb{C}}$ of Gr_{res}^j. For this purpose, we use various equivalent definitions of the complexified orbit $\mathcal{O}_j^{\mathbb{C}}$. To give an explicit formula of the Kähler potential associated with this complex structure, we use an invariant of the $G^{\mathbb{C}}$-orbits. The expression of the potential as a function of the curvature of Gr_{res} obtained by this method is again analogous to the one given by O. Biquard and P. Gauduchon in [4]. An equivalent expression, in terms of characteristic angles of a pair of subspaces $(P, Q) \in \mathcal{O}_j^{\mathbb{C}}$ is also given.

2 Background on Kähler and hyperkähler quotients of Banach manifolds

2.1 Kähler quotient

Let $\mathcal{M}$ be a smooth Banach manifold over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, endowed with a smooth action of a Banach Lie group G (over $\mathbb{K}$), whose Lie algebra will be denoted by $\mathfrak{g}$. For a Banach space B over $\mathbb{K}$, we will denote by B' the topological dual space of B, i.e. the Banach space of continuous linear applications from B to $\mathbb{K}$.

Definition 2.1 A weak symplectic form ω on $\mathcal{M}$ is a closed smooth 2-form on $\mathcal{M}$ such that for all x in $\mathcal{M}$ the map

$$\begin{array}{cccc} \varphi_x : & T_x\mathcal{M} & \to & T'_x\mathcal{M} \\ & X & \mapsto & i_X\omega \end{array}$$

is an injection.

Definition 2.2 A moment map for a G-action on a weakly symplectic Banach manifold $\mathcal{M}$ is a map $\mu : \mathcal{M} \rightarrow \mathfrak{g}'$, satisfying

$$d\mu_x(\mathfrak{a}) = i_{X^{\mathfrak{a}}}\,\omega,$$

for all x in $\mathcal{M}$ and for all $\mathfrak{a}$ in $\mathfrak{g}$, where $X^{\mathfrak{a}}$ denotes the vector field on $\mathcal{M}$ generated by the infinitesimal action of $\mathfrak{a} \in \mathfrak{g}$. The G-action is called *Hamiltonian* if there exists a G-equivariant moment map μ, i.e. a moment map satisfying the following condition :

$$\mu(g \cdot x) = \mathrm{Ad}^*(g)\,(\mu(x))\,.$$

Definition 2.3 A regular value of the moment map is an element $\xi \in \mathfrak{g}'$ such that, for every x in the level set $\mu^{-1}(\xi)$, the map $d\mu_x : T_x\mathcal{M} \rightarrow \mathfrak{g}'$ is surjective and its kernel admits a closed complement in $T_x\mathcal{M}$.

Remark 2.4 If ξ is a regular value of μ, then $\mu^{-1}(\xi)$ is a submanifold of $\mathcal{M}$. If μ is G-equivariant and if ξ is an $\mathrm{Ad}^*(G)$-invariant element of $\mathfrak{g}'$, then the manifold $\mu^{-1}(\xi)$ is globally G-stable, and one can consider the quotient space $\mu^{-1}(\xi)/G$.

In the following, we consider an Hamiltonian action of G on $\mathcal{M}$ and a regular $\mathrm{Ad}^*(G)$-invariant element ξ of $\mathfrak{g}'$. We recall some classical results on the topology and geometry of the quotient space. Propositions 2.5, 2.6 and 2.7 are respectively Proposition 3 chap.III §4.2 in [7], Proposition 6 chap.III §4.3 in [7] and Proposition 10 chap.III §1.5 in [8], up to notational changes.

Proposition 2.5 ([7]) *If G acts properly on a manifold $\mathcal{N}$, then the quotient space $\mathcal{N}/G$ endowed with the quotient topology is Hausdorff.*

Proposition 2.6 ([7]) *If G acts freely on $\mathcal{N}$, the action of G is proper if and only if the graph $\mathcal{C}$ of the equivalence relation defined by G is closed in $\mathcal{N} \times \mathcal{N}$ and the canonical application from $\mathcal{C}$ to G is continuous.*

Proposition 2.7 ([8]) *Assume that G acts freely and properly on $\mathcal{N}$. If, for every $x \in \mathcal{N}$, the tangent space $T_xG\cdot x$ to the orbit $G\cdot x$ at x is closed in $T_x\mathcal{N}$ and admits a closed complement, then the quotient space $\mathcal{N}/G$ has a unique structure of Banach manifold such that the projection $\pi : \mathcal{N} \rightarrow \mathcal{N}/G$ is a submersion.*

Remark 2.8 Let b be a continuous bilinear form on a Banach vector space B. Suppose that b realizes an injection of B into its topological dual B' by $\tilde{b}(X) := b(X, \cdot\,)$, for $X \in B$. For any linear subspace A of B, we have the inclusion

$$\bar{A} \subset \left(A^{\perp_b}\right)^{\perp_b}$$

but not necessarily the equality. The equality means that any continuous linear form vanishing on $A^{\perp_b}$ is of the form $\tilde{b}(X)$ for some $X \in A$, which is a particular property of the subspace A. Along the same lines, if b is a positive definite symmetric bilinear form on B, we have :

$$A \cap A^{\perp_b} = \{0\},$$

but in general we do not have :

$$B = A \oplus A^{\perp_b} \tag{1}$$

even if A is closed, since the right hand side may not be closed. Furthermore J. Lindenstrauss and L. Tzafriri have proved in [26] that a Banach space in which every closed subspace is complemented is isomorphic to a Hilbert space. This implies in particular that for a non-reflexive Banach space B endowed with a weak Riemannian metric, equality (1) is certainly not fulfilled by every closed subspace A.

Proposition 2.9 *If $\mu^{-1}(\xi)/G$ has a Banach manifold structure such that the quotient map is a submersion, and if G acts by symplectomorphisms, the condition*

$$T_xG\!\cdot\! x = \left((T_xG\!\cdot\! x)^{\perp_\omega}\right)^{\perp_\omega} \quad \text{for all} \quad x \in \mu^{-1}(\xi)$$

implies that $\mu^{-1}(\xi)/G$ is a weakly symplectic manifold.

$\square$ **Proof of Proposition** 2.9:
Denote by π the quotient map $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G$. Let us show that the following expression

$$\omega_{red,[x]}(X, Y) := \omega_x(\tilde{X}, \tilde{Y}), \tag{2}$$

where X, Y are in $T_{[x]}(\mu^{-1}(\xi)/G)$ and where $\pi_*\tilde{X} = X$ and $\pi_*\tilde{Y} = Y$, defines a weak symplectic structure on the quotient. Note that for all $x \in \mu^{-1}(\xi)$, the tangent space $T_x(\mu^{-1}(\xi))$ is precisely the kernel of the differential $d\mu_x$, so that for all $\mathfrak{a} \in \mathfrak{g}$, the 1-form $i_{X^{\mathfrak{a}}}\omega$ vanishes on $\mu^{-1}(\xi)$. This implies that the right-hand side of (2) does not depend on the choice of $\tilde{X}$ and $\tilde{Y}$. Since G acts by symplectomorphisms, it does not depend on the choice of the element x in the class $[x]$ either.

It follows that $\pi^*\omega_{red} = \omega_{|\mu^{-1}(\xi)}$. Since ω is closed, so is ω_{red}. The kernel of ω_{red} at a point $[x]$ in the quotient space is :

$$\pi_*\left(T_x\left(\mu^{-1}(\xi)\right)^{\perp_\omega}\right).$$

Note that the tangent space $T_xG{\cdot}x$ to the G-orbit of x is spanned by $\{X^{\mathfrak{a}}(x), \mathfrak{a} \in \mathfrak{g}\}$, hence we have

$$T_x\left(\mu^{-1}(\xi)\right) = (T_xG{\cdot}x)^{\perp_\omega},$$

and

$$\left(T_x\left(\mu^{-1}(\xi)\right)\right)^{\perp_\omega} = \left((T_xG{\cdot}x)^{\perp_\omega}\right)^{\perp_\omega}.$$

For ω_{red} to be symplectic, one needs $T_x\left(\mu^{-1}(\xi)\right)^{\perp_\omega} = T_xG{\cdot}x$, which is precisely the hypothesis. $\square$

Recall the following definition :

Definition 2.10 A G-equivariant slice of the manifold $\mu^{-1}(\xi)$ is a subbundle H of the tangent bundle $T\left(\mu^{-1}(\xi)\right)$ such that, for every x in $\mu^{-1}(\xi)$, H_x is a closed complement to the tangent space $T_xG{\cdot}x$ of the G-orbit $G{\cdot}x$, and such that

$$H_{g\cdot x} = g_*H_x,$$

for all x in $\mu^{-1}(\xi)$ and for all g in G.

Remark 2.11 Suppose that the manifold $\mathcal{M}$ is endowed with a weakly Riemannian G-invariant metric g. Then the existence of a G-invariant slice H of $\mu^{-1}(\xi)$ allows one to define a weakly Riemannian metric g_{red} on the quotient $\mu^{-1}(\xi)/G$, as follows. For every $x \in \mu^{-1}(\xi)$, we set

$$\begin{array}{rcccl} \mathrm{g}_{red,[x]} & : & T_{[x]}\left(\mu^{-1}(\xi)/G\right) \times T_{[x]}\left(\mu^{-1}(\xi)/G\right) & \rightarrow & \mathbb{R} \\ & & (\ X\ ,\ Y\ ) & \mapsto & \mathrm{g}_x(\tilde{X}, \tilde{Y}), \end{array}$$

where $\tilde{X}$ and $\tilde{Y}$ are the unique elements of H_x such that $\pi_*(\tilde{X}) = X$ and $\pi_*(\tilde{Y}) = Y$.

Definition 2.12 A weak Kähler manifold is a Banach manifold $\mathcal{M}$ endowed with a weak symplectic form ω and a weak Riemannian metric g (i.e. such that at every x in $\mathcal{M}$, g defines an injection of $T_x\mathcal{M}$ into its dual), satisfying the following compatibility condition :

(C) the endomorphism I of the tangent bundle of $\mathcal{M}$ defined by $\mathrm{g}(IX, Y) = \omega(X, Y)$ satisfies $I^2 = -1$ and the Nijenhuis tensor N of I vanishes.

Recall that the Nijenhuis tensor has the following expression at x in $\mathcal{M}$:

$$N_x(X,Y) := [X,Y] + I[X,IY] + I[IX,Y] - [IX,IY],$$

where X and Y belong to $T_x\mathcal{M}$.

Theorem 2.13 *Let $\mathcal{M}$ be a smooth Kähler Banach manifold endowed with a free and proper Hamiltonian action of a Banach Lie group G preserving the Kähler structure. If, for every x in the preimage $\mu^{-1}(\xi)$ of an $Ad^*(G)$-invariant regular value of the moment map μ, the tangent space $T_xG\cdot x$ of the orbit $G\cdot x$ satisfies the direct sum condition*

$$\text{(D)} \qquad T_xG{\cdot}x \;\oplus\; (T_xG{\cdot}x)^{\perp_{\mathrm{g}}} \;=\; T_x\left(\mu^{-1}(\xi)\right),$$

then the quotient space $\mathcal{M}//G := \mu^{-1}(\xi)/G$ is a smooth Kähler manifold.

■ **Proof of Theorem** 2.13:
Let us denote by (g,ω,I) the Kähler structure of $\mathcal{M}$. For x in $\mu^{-1}(\xi)$, the condition

$$T_xG{\cdot}x \oplus (T_xG{\cdot}x)^{\perp_{\mathrm{g}}} = T_x\mu^{-1}(\xi),$$

implies that the tangent space of the orbit $G{\cdot}x$ satisfies the following property :

$$\left((T_xG{\cdot}x)^{\perp_{\mathrm{g}}}\right)^{\perp_{\mathrm{g}}} = T_xG{\cdot}x \tag{3}$$

(the converse may not be true). In particular, $T_xG{\cdot}x$ is closed and splits, so one is able to define on the quotient space a Banach manifold structure by use of Proposition 2.7. Now equality (3) is equivalent to

$$\left((T_xG{\cdot}x)^{\perp_{\omega}}\right)^{\perp_{\omega}} = T_xG{\cdot}x,$$

since I is orthogonal with respect to g. So the condition needed for the definition of the symplectic structure on the quotient in Proposition 2.9 is fulfilled. Moreover the orthogonal $H_x := (T_xG{\cdot}x)^{\perp_{\mathrm{g}}}$ of $T_xG{\cdot}x$ in $T_x\left(\mu^{-1}(\xi)\right)$ defines a G-equivariant slice for the manifold $\mu^{-1}(\xi)$, and, by Remark 2.10, allows one to define a Riemannian metric on the quotient. It remains to define a compatible complex structure I_{red} on $\mu^{-1}(\xi)/G$. For this purpose, let us remark that

$$IH_x = I\,(T_xG{\cdot}x)^{\perp_{\mathrm{g}}} \subset T_x\mu^{-1}(\xi) = (T_xG{\cdot}x)^{\perp_{\omega}}\,,$$

since $\omega(X^{\mathfrak{a}}, IU) = \mathrm{g}(IX^{\mathfrak{a}}, IU) = \mathrm{g}(X^{\mathfrak{a}}, U) = 0$ for all $\mathfrak{a}$ in $\mathfrak{g}$ and U in H_x. In addition, IH_x is orthogonal to $T_xG{\cdot}x$ with respect to g since $\mathrm{g}(IU, X^{\mathfrak{a}}) = \omega(U, X^{\mathfrak{a}}) = 0$ for $U \in T_x\left(\mu^{-1}(\xi)\right) = (T_x G\cdot x)^{\perp_\omega}$. This implies that H_x is stable under I and one can define a complex structure I_{red} on the quotient space by:

$$\begin{array}{rcccl} I_{red} & : & T_{[x]}\left(\mu^{-1}(\xi)/G\right) & \rightarrow & T_{[x]}\left(\mu^{-1}(\xi)/G\right) \\ & & X & \mapsto & \pi_* I_x \tilde{X}, \end{array}$$

where x is in $[x]$ and where $\tilde{X}$ is the unique element of H_x whose projection on $T_{[x]}\left(\mu^{-1}(\xi)/G\right)$ is X. Hence one has :

$$I_{red}\pi_*\tilde{X} = \pi_* I\tilde{X}.$$

The application π being a submersion, given two vector fields X and Y on $\mu^{-1}(\xi)/G$, one has :

$$[X, Y] = \pi_*\left([\tilde{X}, \tilde{Y}]\right),$$

where again $\tilde{X}$ satisfies $\tilde{X}(x) \in H_x$ and $\pi_*(\tilde{X}) = X$, and similar conditions for $\tilde{Y}$. Therefore the formal integrability condition on I implies the formal integrability condition on I_{red} since the Nijenhuis tensor of I_{red} has the following expression:

$$\begin{array}{rl} N(X,Y) & := [X,Y] + I_{red}[X, I_{red}Y] + I_{red}[I_{red}X, Y] - [I_{red}X, I_{red}Y] \\ & = \pi_*[\tilde{X},\tilde{Y}] + I_{red}\pi_*[\tilde{X}, I\tilde{Y}] + I_{red}\pi_*[I\tilde{X},\tilde{Y}] - \pi_*[I\tilde{X}, I\tilde{Y}] \\ & = \pi_*(N(\tilde{X},\tilde{Y})). \end{array}$$

where X, Y are in $T_{[x]}\left(\mu^{-1}(\xi)/G\right)$ and $\tilde{X}, \tilde{Y}$ are as before. ■

Remark 2.14 In contrast to the finite-dimensional case illustrated by the Newlander-Nirenberg Theorem (see [34]), the formal integrability condition of an almost complex structure I on a Banach manifold $\mathcal{M}$ given by a vanishing Nijenhuis tensor is not sufficient for $\mathcal{M}$ to admit a system of holomorphic charts. An example of a formally integrable complex structure on a real Banach manifold which does not admit any open subset biholomorphic to an open subset of a complex Banach manifold was recently constructed by I. Patyi in [35]. However, if $\mathcal{M}$ is a real analytic manifold and I a formally integrable analytic complex structure, then $\mathcal{M}$ can be endowed with a holomorphic atlas (see [36], and [1] for the details of this result). Note also that, in the Fréchet context, L. Lempert showed in [25] that the complex structure defined in [31] by J. E. Marsden and A. Weinstein on the space of knots does not lead to the existence of holomorphic charts, although this structure was shown to be formally integrable by J. L. Brylinski in [10]. In the context of formally integrable complex structures, we will call a map f between two complex manifolds (M, I_M) and (N, I_N) *holomorphic* if $df \circ I_M = I_N \circ df$.

2.2 Stable manifold

Let $(\mathcal{M}, \omega, \mathrm{g}, I)$ be a smooth Kähler Banach manifold endowed with a smooth Hamiltonian action of a Banach Lie group G preserving the Kähler structure. Let ξ be an $\mathrm{Ad}^*(G)$-invariant regular value of the moment map μ. Assume that there exists a complex Lie group $G^{\mathbb{C}}$ with Lie algebra $g^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$ which acts holomorphically and smoothly on $\mathcal{M}$ extending the action of G, and that the following assumption holds :

(H) for every x in $\mu^{-1}(\xi)$, one has: $T_x\mathcal{M} \;=\; T_x\left(\mu^{-1}(\xi)\right) \oplus I\, T_xG{\cdot}x$ as topological sum.

(Note that, by definition of the moment map, one has $T_x\left(\mu^{-1}(\xi)\right) \;=\; (I\; T_xG\cdot x)^{\perp_{\mathrm{g}}}$, so (H) states that the direct sum of $I\, T_xGx$ and its orthogonal is closed in $T_x\mathcal{M}$, which is not always the case as mentioned in Remark 2.8, but nevertheless a natural assumption to make.) The action of $G^{\mathbb{C}}$ on $\mathcal{M}$ allows one to define a notion of stable manifold associated with the level set $\mu^{-1}(\xi)$:

Definition 2.15 *The stable manifold $\mathcal{M}^s$ associated with the level set $\mu^{-1}(\xi)$ is defined by:*

$$\mathcal{M}^s \;:= \{x \in \mathcal{M} \;\;|\;\; \exists\, g \in G^{\mathbb{C}},\;\; g\cdot x \in \mu^{-1}(\xi)\}.$$

Remark 2.16 The assumption (H) implies that $\mathcal{M}^s$ is open in $\mathcal{M}$ since $T_x\mathcal{M}^s = T_x\mathcal{M}$ for every element x in $\mu^{-1}(\xi)$, hence, by translation by an element of $G^{\mathbb{C}}$, for every x in $\mathcal{M}^s$.

Proposition 2.17 *If $G^{\mathbb{C}}$ admits a polar decomposition $G^{\mathbb{C}} = \exp i\mathfrak{g}\cdot G$, then for every x in $\mu^{-1}(\xi)$, one has:*

$$G^{\mathbb{C}}{\cdot}x \;\cap\; \mu^{-1}(\xi) \;\;=\;\; G{\cdot}x.$$

□ **Proof of Proposition** 2.17:
Clearly $G{\cdot}x \;\subset\; G^{\mathbb{C}}{\cdot}x \cap \mu^{-1}(\xi)$ since ξ is $\mathrm{Ad}^*(G)$-invariant. Let us show that $G^{\mathbb{C}}{\cdot}x \cap \mu^{-1}(\xi) \;\subset\; G{\cdot}x$. Suppose that there exists $g \in G^{\mathbb{C}}$ such that $g\cdot x \in \mu^{-1}(\xi)$. Since $\mu^{-1}(\xi)$ is G-invariant and since $G^{\mathbb{C}} = \exp i\mathfrak{g}{\cdot}G$, it is sufficient to consider the case when $g = \exp i\mathfrak{a}$, $\mathfrak{a} \in \mathfrak{g}$.

Define the function $h\; : \mathbb{R} \to \mathbb{R}$ by $h(t) = \mu\left((\exp it\mathfrak{a})\cdot x\right)(\mathfrak{a})$. One has $h(0) = h(1) = \xi(\mathfrak{a})$, hence there exists $t_0 \in\; (0, 1)$ such that :

$$0 = h'(t_0) = d_y\mu(i\mathfrak{a}\cdot y)(\mathfrak{a}) = -\omega_y(i\mathfrak{a}\cdot y, \mathfrak{a}\cdot y) = \|\mathfrak{a}\cdot y\|^2,$$

where $y = \exp(it_0\mathfrak{a})\cdot x$. Hence $\mathfrak{a}\cdot y = 0$ and $\exp(i\mathfrak{a}\mathbb{R})$ fixes y, thus also x. It follows that :

$$\exp(i\mathfrak{a}\mathbb{R})\cdot x \;\cap\; \mu^{-1}(\xi) \;\;=\;\; \{x\}.$$

$\square$

From now on and till the end of Section 2, it will be assumed that $G^{\mathbb{C}}$ admits a polar decomposition.

Corollary 2.18 *If G acts freely on $\mu^{-1}(\xi)$, then $G^{\mathbb{C}}$ acts freely on $\mathcal{M}^s$.*

$\square$ **Proof of Corollary** 2.18:
Let x be an element of $\mu^{-1}(\xi)$ and let $g \in G^{\mathbb{C}}$ be such that $g \cdot x = x$. Since $G^{\mathbb{C}} = \exp i\mathfrak{g} \cdot G$, there exists $u \in G$ and $\mathfrak{a} \in \mathfrak{g}$ such that $g = \exp(i\mathfrak{a})u$, and one has $\exp(i\mathfrak{a})u \cdot x = x$. From the proof of the previous Proposition , one has:

$$\exp(i\mathfrak{a}\mathbb{R}).(ux) \ \cap \ \mu^{-1}(\xi) \ = \ \{ux\}.$$

It follows that $ux = x$, thus $u = e$, since G acts freely on $\mu^{-1}(\xi)$. Now, the condition $\exp(i\mathfrak{a}\mathbb{R}) \cdot x = x$ implies that $\mathfrak{a}$ fixes x, hence $\mathfrak{a} = 0$. $\square$

Proposition 2.19 *Assume that G acts freely on $\mu^{-1}(\xi)$. Then, for every y in $\mathcal{M}^s$, there is a unique element $g(y)$ in $\exp i\mathfrak{g}$ such that $g(y)$ maps y to the level set. The resulting application*

$$\begin{array}{rccc} g & : & \mathcal{M}^s & \rightarrow & \exp i\mathfrak{g} \\ & & y & \mapsto & g(y) \end{array}$$

is smooth and the projection q

$$\begin{array}{rccc} q & : & \mathcal{M}^s & \rightarrow & \mu^{-1}(\xi) \\ & & y & \mapsto & g(y) \cdot y \end{array}$$

is smooth and G-equivariant.

$\square$ **Proof of Proposition** 2.19 :
. Let y be in $\mathcal{M}^s$ and suppose that there exist two elements $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathfrak{g}$ such that both $\exp i\mathfrak{a} \cdot y$ and $\exp i\mathfrak{b} \cdot y$ belong to $\mu^{-1}(\xi)$. Since $\exp i\mathfrak{a} \cdot y$ and $\exp i\mathfrak{b} \cdot y$ are in the same $G^{\mathbb{C}}$-orbit, by Proposition 2.17, there exists u in G such that $\exp i\mathfrak{a} \cdot y = u \cdot \exp i\mathfrak{b} \cdot y$. Since G acts freely on the level set, Corollary 2.18 implies that $G^{\mathbb{C}}$ acts freely on $\mathcal{M}^s$. It follows that $\exp i\mathfrak{a} = u \cdot \exp i\mathfrak{b}$, hence, by uniqueness of the polar decomposition, u is the unit element of G and $\exp i\mathfrak{a} = \exp i\mathfrak{b}$. Therefore $g(y) = \exp i\mathfrak{a}$ is well defined, and so is the projection q.
. Let us show that the application :

$$\begin{array}{rccc} g & : & \mathcal{M}^s & \rightarrow & \exp i\mathfrak{g} \\ & & y & \mapsto & g(y) \end{array}$$

is smooth. Since $G^{\mathbb{C}}$ acts smoothly on $\mathcal{M}^s$, it will imply that the projection q is smooth also. Consider the following map :

$$\begin{array}{rccl} \phi : & \exp i\mathfrak{g} \times \mathcal{M}^s & \to & \mathcal{M} \\ & (\ \exp i\mathfrak{a}\ ,\ y\ ) & \mapsto & \exp i\mathfrak{a} \cdot y, \end{array}$$

which maps $\exp i\mathfrak{g} \times \mathcal{M}^s$ onto $\mathcal{M}^s$. (Recall that $\exp i\mathfrak{g}$ inherits a Banach manifold structure from its identification with the homogeneous space $G^{\mathbb{C}}/G$.) We will prove that ϕ is transversal to $\mu^{-1}(\xi)$ (see §5.11.6 and §5.11.7 of [9] for a definition of this notion), so that the subset

$$\phi^{-1}\left(\mu^{-1}(\xi)\right) \ = \ \{\ (g(y)\ ,\ y)\ ,\ y\ \in\ \mathcal{M}^s\ \}$$

is a smooth submanifold of $\exp i\mathfrak{g} \times \mathcal{M}^s$. The smoothness of the application g will therefore follow from the smoothness of the projection $p_1\ : \exp i\mathfrak{g} \times \mathcal{M}^s \to \exp i\mathfrak{g}$ on the first factor. We will denote by R_y the right translation by y on $G^{\mathbb{C}}$. The differential of ϕ at a point $(\exp i\mathfrak{a}, y)$ in $\exp i\mathfrak{g} \times \mathcal{M}^s$ reads :

$$(d\phi)_{(\exp i\mathfrak{a}, y)}\left(((R_{\exp i\mathfrak{a}})_*(i\mathfrak{b}), Z)\right) := i\mathfrak{b} \cdot (\exp i\mathfrak{a} \cdot y) \oplus (\exp i\mathfrak{a})_*(Z).$$

Note that, for every element $(\exp i\mathfrak{a}, y)$ in $\phi^{-1}\left(\mu^{-1}(\xi)\right)$, one has :

$$(d\phi)_{(\exp i\mathfrak{a}, y)}\left(\{0\} \times T_y\mathcal{M}^s\right) \ = \ T_x\mathcal{M}^s$$

where $x := \exp i\mathfrak{a} \cdot y$, so that $(d\phi)_{(\exp i\mathfrak{a}, y)}$ is surjective. It remains to show that the subspace

$$(d\phi)^{-1}_{(\exp i\mathfrak{a}, y)}\left(T_x(\mu^{-1}(\xi))\right)$$

is complemented. For this purpose recall that, by assumption (H), the tangent space $T_y\mathcal{M}^s$ is isomorphic to

$$(\exp(-i\mathfrak{a}))_*\left(T_x\left(\mu^{-1}(\xi)\right)\right) \oplus\ (\exp(-i\mathfrak{a}))_*\left(i\mathfrak{g} \cdot x\right),$$

so that the tangent space

$$T_{(\exp i\mathfrak{a}, y)}\left(\exp i\mathfrak{g} \times \mathcal{M}^s\right) \ = \ T_{\exp i\mathfrak{a}}\left(\exp i\mathfrak{g}\right) \times T_y\mathcal{M}^s$$

is isomorphic to $\mathfrak{g} \times T_x\left(\mu^{-1}(\xi)\right) \times \mathfrak{g}$ by the following isomorphism :

$$\begin{array}{rccl} \jmath : & \mathfrak{g} \times T_x\left(\mu^{-1}(\xi)\right) \times \mathfrak{g} & \to & T_{\exp i\mathfrak{a}}\left(\exp i\mathfrak{g}\right) \times T_y\mathcal{M}^s \\ & (\ \mathfrak{b}\ ,\ W\ ,\ \mathfrak{c}\ ) & \mapsto & \left((R_{\exp i\mathfrak{a}})_*(i\mathfrak{b})\ ,\ (\exp(-i\mathfrak{a}))_*(W) + (\exp(-i\mathfrak{a}))_*(i\mathfrak{c} \cdot x)\right). \end{array}$$

The element $\jmath(\mathfrak{b}, W, \mathfrak{c})$ belongs to $(d\phi)^{-1}_{(\exp i\mathfrak{a}, y)}$ whenever $i\mathfrak{b} \cdot x + W + i\mathfrak{c} \cdot x \in T_x\left(\mu^{-1}(\xi)\right)$. Since G acts freely on $\mathcal{M}$, it follows that the subspace :

$$(d\phi)^{-1}_{(\exp i\mathfrak{a}, y)}\left(T_x(\mu^{-1}(\xi))\right)$$

equals

$$\{\ \jmath(\mathfrak{b}, W, -\mathfrak{b})\ ,\ \mathfrak{b} \in \mathfrak{g}, W \in T_x\left(\mu^{-1}(\xi)\right)\ \},$$

and

$$\{\ \jmath(\mathfrak{b}, 0, \mathfrak{b})\ ,\ \mathfrak{b} \in \mathfrak{g}\ \}$$

is a closed complement to it.

. Let us check the G-equivariance of q. Since μ is G-equivariant and ξ is $\mathrm{Ad}^*(G)$-invariant, one has :

$$\mu(u \cdot g(y) \cdot y) = (Ad)^*(u)\left(\mu(g(y) \cdot y)\right) = (Ad)^*(u)(\xi) = \xi$$

for all u in G, and y in $\mathcal{M}^s$. We can write $g(y) = \exp i\mathfrak{a}$ for some $\mathfrak{a} \in \mathfrak{g}$. Now the equality $u \cdot \exp i\mathfrak{a} = \exp\left(\mathrm{Ad}(u)(i\mathfrak{a})\right) \cdot u$ and the uniqueness of the element $g(u \cdot y)$ satisfying $g(u \cdot y) \cdot (u \cdot y)$ proved above, imply that :

$$g(u \cdot y)\ =\ \exp\left(\mathrm{Ad}(u)(i\mathfrak{a})\right).$$

Hence q satisfies the G-equivariant condition :

$$q(u \cdot y)\ =\ u \cdot q(y)$$

for all $u \in G$ and $y \in \mathcal{M}^s$. □

Proposition 2.20 *If G acts freely and properly on $\mu^{-1}(\xi)$, then $G^{\mathbb{C}}$ acts (freely and) properly on $\mathcal{M}^s$.*

□ **Proof of Proposition** 2.20:
By Proposition 2.18, $G^{\mathbb{C}}$ acts freely on $\mathcal{M}^s$. By Proposition 2.6, $G^{\mathbb{C}}$ acts properly on $\mathcal{M}^s$ if and only if the graph $\tilde{\mathcal{C}}$ of the equivalence relation defined by $G^{\mathbb{C}}$ is closed in $\mathcal{M}^s \times \mathcal{M}^s$ and the canonical map from $\tilde{\mathcal{C}}$ to $G^{\mathbb{C}}$ is continuous.

Let us show that $\tilde{\mathcal{C}}$ is closed in $\mathcal{M}^s \times \mathcal{M}^s$. Denote by $\mathcal{C}$ the graph of the equivalence relation defined by the action of G on $\mu^{-1}(\xi)$. Let $\{(y_n, v_n \cdot y_n)\}_{n\in\mathbb{N}}$ be a sequence in $\tilde{\mathcal{C}}$, where $y_n \in \mathcal{M}^s$ and $v_n \in G^{\mathbb{C}}$, which converges to an element (y_∞, z_∞) in $\mathcal{M}^s \times \mathcal{M}^s$. From Proposition 2.17 and from the continuity of the projection q, the sequence $\{(q(y_n), q(v_n \cdot y_n))\}_{n\in\mathbb{N}}$ belongs to $\mathcal{C}$ and converges to $(q(y_\infty), q(z_\infty))$. Since $\mathcal{C}$ is closed in $\mu^{-1}(\xi) \times \mu^{-1}(\xi)$, it follows that $q(z_\infty) = u_\infty \cdot q(y_\infty)$ for some u_∞ in G. Hence $z_\infty = g(z_\infty)^{-1} u_\infty g(y_\infty) \cdot y_\infty$. Thus $\tilde{\mathcal{C}}$ is closed in $\mathcal{M}^s \times \mathcal{M}^s$.

Let us show that the canonical map $\tilde{\iota}$ from $\tilde{\mathcal{C}}$ to $G^{\mathbb{C}}$ is continuous. Denote by ι the canonical map from $\mathcal{C}$ to G. One has :

$$\tilde{\iota}(y, z)\ \mapsto\ g(z)^{-1} \circ \iota(q(y), q(z)) \circ g(y),$$

and the continuity of $\tilde{\iota}$ follows from the continuity of the applications ι, g and q. □

We conclude this Subsection with

Theorem 2.21 *Let $\mathcal{M}$ be a Banach Kähler manifold endowed with a smooth, free and proper Hamiltonian action of a Banach Lie group G, which preserves the Kähler structure and extends to a smooth holomorphic action of a complex Lie group $G^{\mathbb{C}} = \exp i\mathfrak{g}\cdot G$. Let ξ be an $Ad^*(G)$-invariant regular value of the moment map μ. If for every x in $\mu^{-1}(\xi)$, the orthogonal $(T_xG\cdot x)^{\perp_g}$ of $T_xG\cdot x$ in $T_x\left(\mu^{-1}(\xi)\right)$ satisfies the following direct sum condition*

$$(\mathrm{D}+\mathrm{H}) \qquad T_x\mathcal{M} \;=\; T_xG\cdot x \;\oplus\; (T_xG\cdot x)^{\perp_g} \;\oplus\; I\left(T_xG\cdot x\right),$$

then the quotient space $\mathcal{M}^s/G^{\mathbb{C}}$ is a smooth complex manifold isomorphic to the smooth Kähler quotient $\mathcal{M}//G := \mu^{-1}(\xi)/G$ as complex smooth manifold. Moreover the full integrability of the complex structure on $\mathcal{M}$ implies the full integrability of the complex structure on $\mathcal{M}^s/G^{\mathbb{C}}$, hence on $\mathcal{M}//G$.

■ **Proof of Theorem** 2.21 :
By Proposition 2.18 and Proposition 2.20, $G^{\mathbb{C}}$ acts freely and properly on $\mathcal{M}^s$. The smoothness of the application g and the G-equivariant slice H of $\mu^{-1}(\xi)$ given by $H_x := (T_xG\cdot x)^{\perp_{\mathrm{g}}}$, allows one to define a G-equivariant slice on $\mathcal{M}^s$, also denoted by H, by the following formula :

$$H_y := g(y)_*^{-1}\left(H_{q(y)}\right) \subset T_y\mathcal{M}^s,$$

for all y in $\mathcal{M}^s$. The following decomposition of the tangent space $T_y\mathcal{M}^s$ holds :

$$T_y\mathcal{M}^s = H_y \oplus T_y\left(G^{\mathbb{C}}\cdot y\right).$$

By Proposition 2.7, it follows that $\mathcal{M}^s/G^{\mathbb{C}}$ has a unique (real) Banach manifold structure such that the quotient map is a submersion. Since for every element y in $\mathcal{M}^s$, $H_{q(y)}$ is invariant under the complex structure I of $\mathcal{M}$, and since the $G^{\mathbb{C}}$-action on $\mathcal{M}$ is holomorphic, for every y in $\mathcal{M}^s$ the subspace H_y of $T_y\mathcal{M}^s$ is I-invariant and, further, $\mathcal{M}^s/G^{\mathbb{C}}$ inherits a natural complex structure. Moreover, since the complex structure of the quotient $\mathcal{M}^s/G^{\mathbb{C}}$ comes from the complex structure of $\mathcal{M}$, the natural injection $\mu^{-1}(\xi) \hookrightarrow \mathcal{M}^s$ induces a complex isomorphism between $\mu^{-1}(\xi)/G$ and $\mathcal{M}^s/G^{\mathbb{C}}$. Finally, $\mathcal{M}^s$ being an open subset of $\mathcal{M}$ because of (H), the existence of holomorphic charts on $\mathcal{M}$ allow one to apply Proposition 2.7 to the holomorphic quotient $\mathcal{M}^s/G^{\mathbb{C}}$, thus implies the existence of holomorphic charts on $\mathcal{M}^s/G^{\mathbb{C}}$. ■

2.3 Hyperkähler quotient

Let $\mathcal{M}$ be a Banach manifold endowed with a weak hyperkähler metric g, with Kähler forms ω_1, ω_2 and ω_3, and corresponding complex structures I_1, I_2, and I_3. Let G be a connected Banach Lie group with Lie algebra $\mathfrak{g}$, acting freely on $\mathcal{M}$ by hyperkähler diffeomorphisms. We assume in this Subsection that there exists a G-equivariant moment map μ_i for each symplectic structure. Define $\mu\ :\mathcal{M}\rightarrow\mathfrak{g}'\otimes\mathbb{R}^3$ by $\mu=\mu_1\oplus\mu_2\oplus\mu_3$. Let $\xi=(\xi_1,\xi_2,\xi_3)$ be a regular $\mathrm{Ad}^*(G)$-invariant value of the moment map μ.

Theorem 2.22 *If G acts freely and properly on $\mathcal{M}$ and if, for every x in $\mu^{-1}(\xi)$, the orthogonal $(T_xG\cdot x)^{\perp_g}$ of $T_xG\cdot x$ in $T_x\left(\mu^{-1}(\xi)\right)$ satisfies the direct sum condition*

$$\text{(D)}\qquad T_xG\cdot x\ \oplus\ (T_xG\cdot x)^{\perp_{\mathrm{g}}}\ =\ T_x\left(\mu^{-1}(\xi)\right),$$

then the quotient space $\mu^{-1}(\xi)/G$ carries a structure of smooth Banach hyperkähler manifold.

■ **Proof of Theorem** 2.22:
This is a direct application of Theorem 2.13 with respect to each complex structure I_1, I_2, and I_3 of $\mathcal{M}$. By the very definition of the complex structures and symplectic forms on the reduced space, $\mu^{-1}(\xi)/G$ inherits a hyperkähler structure from the hyperkähler structure of $\mathcal{M}$.
■

Let us now assume that the orthogonal $(T_xG\cdot x)^{\perp_{\mathrm{g}}}$ of T_xGx in $T_x\left(\mu^{-1}(\xi)\right)$ satisfies the direct sum condition

(S) for every x in $\mu^{-1}(\xi)$, one has: $T_x\mathcal{M}\ =\ T_xGx\oplus(T_xG\cdot x)^{\perp_{\mathrm{g}}}\oplus I_1T_xG\cdot x\ \oplus\ I_2T_xG\cdot x\ \oplus\ I_3T_xG\cdot x$ as a topological direct sum.

For each complex structure $I_{\vec{k}}=k_1\,I_1+k_2\,I_2+k_3\,I_3$ in the 2-sphere of complex structures on $\mathcal{M}$, indexed by $\vec{k}=(k_1,k_2,k_3)$ in S^2, let us define an action $\cdot_{\vec{k}}$ of $i\mathfrak{g}$ on $\mathcal{M}$ by :

$$i\mathfrak{a}\cdot_{\vec{k}}x=I_{\vec{k}}(\mathfrak{a}\cdot x),$$

for all $\mathfrak{a}$ in $\mathfrak{g}$ and for all x in $\mathcal{M}$. Assume that for a given $\vec{k}$, the action $\cdot_{\vec{k}}$ integrates into an $I_{\vec{k}}$-holomorphic action of $G^{\mathbb{C}}$ on $\mathcal{M}$. Let us choose an orthogonal complex structure to $I_{\vec{k}}$ denoted by $I_{\vec{l}}$, and define a third complex structure by $I_{\vec{m}}:=I_{\vec{k}}.I_{\vec{l}}$, so that $\left(I_{\vec{k}}\,,\,I_{\vec{l}},\,I_{\vec{m}}\right)$ satisfies the quaternionic identities. Denote by $\mu_{\vec{k}}$ the combination $\mu_{\vec{k}}:=k_1\,\mu_1+k_2\,\mu_2+k_3\,\mu_3$ and similarly $\mu_{\vec{l}}$ and $\mu_{\vec{m}}$. Consider $\xi_{\vec{l}}:=l_1\,\xi_1+l_2\,\xi_2+l_3\,\xi_3$ and $\xi_{\vec{m}}\ :=\ m_1\,\xi_1+m_2\,\xi_2+m_3\,\xi_3$, and similarly $\omega_{\vec{l}}:=l_1\,\omega_1+l_2\,\omega_2+l_3\,\omega_3$ and $\omega_{\vec{m}}:=m_1\,\omega_1+m_2\,\omega_2+m_3\,\omega_3$.

Proposition 2.23 *The map $\mu^{\mathbb{C}}:=\mu_{\vec{l}}+i\mu_{\vec{m}}$ is an $Ad^*\left(G^{\mathbb{C}}\right)$-equivariant holomorphic moment map for the $I_{\vec{k}}$-complex symplectic structure $\omega_{\vec{l}}+i\omega_{\vec{m}}$.*

$\square$ **Proof of Proposition** 2.23:
By the same algebraic arguments than the ones given in [14], section 3 (D), the map $\mu^{\mathbb{C}}$ is holomorphic and a moment map for $\omega_{\vec{l}} + i\omega_{\vec{m}}$. Since G is connected, the $\mathrm{Ad}^*\left(G^{\mathbb{C}}\right)$-equivariance of $\mu^{\mathbb{C}}$ is will follow from

$$\langle d\mu^{\mathbb{C}}(i\mathfrak{a}.x), \mathfrak{b}\rangle = \langle \mu^{\mathbb{C}}(x), [\mathfrak{a}, \mathfrak{b}]\rangle,$$

for all x in $\mathcal{M}$, and all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{g}$. But this is an easy consequence of the $\mathrm{Ad}^*(G)$-equivariance of $\mu_{\vec{l}}$ and $\mu_{\vec{m}}$ and the following identities

$$d\mu^{\mathbb{C}}(i\mathfrak{a}.x) := d\mu^{\mathbb{C}}(I_1 X^{\mathfrak{a}}) = i d\mu^{\mathbb{C}}(X^{\mathfrak{a}}) \quad \text{and} \quad \mu^{\mathbb{C}}(x)\left([i\mathfrak{a}, \mathfrak{b}]\right) = i\mu^{\mathbb{C}}(x)\left([\mathfrak{a}, \mathfrak{b}]\right),$$

where $X^{\mathfrak{a}}$ denotes the vector field generated by $\mathfrak{a} \in \mathfrak{g}$. $\square$

Lemma 2.24 *The stable manifold $\mathcal{M}^{s_{\vec{k}}}$ with respect to the complex structure $I_{\vec{k}}$, associated with the level set $\mu^{-1}(\xi)$, is a submanifold of the Banach manifold $\mathcal{M}$, contained in the preimage by $\mu^{\mathbb{C}}$ of the $G^{\mathbb{C}}$-coadjoint of $\xi_{\vec{l}} + i\xi_{\vec{m}}$. In particular, if $\xi_{\vec{l}} + i\xi_{\vec{m}}$ is in the center of $\mathfrak{g}^{\mathbb{C}}$, then $\mathcal{M}^{s_{\vec{k}}} \subset \mu_{\vec{l}}^{-1}(\xi_{\vec{l}}) \ \cap \ \mu_{\vec{m}}^{-1}(\xi_{\vec{m}})$.*

$\triangle$ **Proof of Lemma** 2.24:
The fact that $\mathcal{M}^{s_{\vec{k}}}$ is included in $\left(\mu^{\mathbb{C}}\right)^{-1}\left(\mathrm{Ad}^*\left(G^{\mathbb{C}}\right)\left(\xi_{\vec{l}} + i\xi_{\vec{m}}\right)\right)$ is a direct consequence of the $\mathrm{Ad}^*(G^{\mathbb{C}})$-equivariance of $\mu^{\mathbb{C}}$ (Proposition 2.23). Let x be an element in the level set $\mu^{-1}(\xi)$. Since ξ is a regular value of μ, the level set $\mu^{-1}(\xi)$ is a Banach submanifold of $\mathcal{M}$. Consider an adapted chart $(\mathcal{V}, E \times F, \varphi)$ of the submanifold $\mu^{-1}(\xi)$ at x, where $\mathcal{V}$ is a neighborhood of x in $\mathcal{M}$, E and F are two Banach spaces, and φ is an homeomorphism from $\mathcal{V}$ onto a neighborhood of 0 in $E \times F$ such that $\mathcal{U}_1 := \varphi\left(\mu^{-1}(\xi) \cap \mathcal{V}\right) \subset E$. By assumption (S), one has :

$$T_x\mathcal{M} \;=\; T_xGx \oplus (T_xG\cdot x)^{\perp_{\mathrm{g}}} \oplus I_{\vec{k}}\left(T_xG\cdot x\right) \oplus I_{\vec{l}}(T_xG\cdot x) \oplus I_{\vec{m}}\left(T_xG\cdot x\right),$$

hence F is isomorphic (as Banach space) to $I_{\vec{k}}\left(T_xG\cdot x\right) \oplus I_{\vec{l}}(T_xG\cdot x) \oplus I_{\vec{m}}\left(T_xG\cdot x\right)$. For $\mathfrak{a}$ in the Lie algebra $\mathfrak{g}$ of G, denote by $X^{\mathfrak{a}}$ the vector field on $\mathcal{M}$ generated by $\mathfrak{a}$. Since G acts freely on $\mathcal{M}$, the map $\mathfrak{g} \to I_{\vec{k}}\left(T_xG\cdot x\right)$ which assigns to $\mathfrak{a} \in \mathfrak{g}$ the vector $I_{\vec{k}}X^{\mathfrak{a}}$ is a continuous bijection of Banach spaces, hence an isomorphism, and similarly for the indexes $\vec{l}$ and $\vec{m}$. Since by hypothesis the G-action on $\mathcal{M}$ extend to a $I_{\vec{k}}$-holomorphic action of $G^{\mathbb{C}}$ on $\mathcal{M}$, the vector fields $I_{\vec{k}}X^{\mathfrak{a}}$, for $\mathfrak{a} \in \mathfrak{g}$, are complete, whereas the flows $t \mapsto f^t_{I_{\vec{l}}X^{\mathfrak{b}} + I_{\vec{m}}X^{\mathfrak{c}}}$ of the vector fields $\left(I_{\vec{l}}X^{\mathfrak{b}} + I_{\vec{m}}X^{\mathfrak{c}}\right)$, with $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathfrak{g}$, may not be globally defined. Nevertheless, by the smooth dependance of the solutions of a differential equation with respect to a parameter, there exists a small neighborhood $\mathcal{U}_2$ of 0 in $I_{\vec{l}}(T_xG\cdot x) \ \oplus \ I_{\vec{m}}\,(T_xG\cdot x) \simeq \mathfrak{g}\oplus\mathfrak{g}$ for which the flows $f^t_{I_{\vec{l}}X^{\mathfrak{b}}+I_{\vec{m}}X^{\mathfrak{c}}}$ are defined for $t\in[0,1]$. Now consider the following map :

$$\begin{array}{rccl}\Psi & : & \mathcal{U}_1\oplus\mathfrak{g}\oplus\mathcal{U}_2 & \rightarrow & \mathcal{M}\\ & & (u,\mathfrak{a},(\mathfrak{b},\mathfrak{c})) & \mapsto & \exp ia\cdot_{\vec{k}} f^1_{I_{\vec{l}}X^{\mathfrak{b}}+I_{\vec{m}}X^{\mathfrak{c}}}\left(\varphi^{-1}(u)\right).\end{array}$$

Then Ψ provides a chart in the neighborhood of any y in the fiber $q_{\vec{k}}^{-1}(x)$ where $q_{\vec{k}}\ : \mathcal{M}^{s_{\vec{k}}}\rightarrow\mu^{-1}(\xi)$ is the projection defined in Proposition 2.19. $\triangle$

With the notation above, Theorem 2.21 reads:

Theorem 2.25 *If G acts freely and properly on $\mathcal{M}$, and if, for every x in $\mu^{-1}(\xi)$, the orthogonal of $T_xG\cdot x$ in $T_x\mu^{-1}(\xi)$ satisfies the direct sum conditions* (D) *and* (S)*, then the quotient space $\mathcal{M}^{s_{\vec{k}}}/G^{\mathbb{C}}$ is a smooth $I_{\vec{k}}$-complex manifold which is diffeomorphic to $\mu^{-1}(\xi)/G$ as a $I_{\vec{k}}$-complex smooth manifold.* ■

2.4 Kähler potential on a Kähler quotient

This Subsection is a generalization to the Banach setting of results obtained in the finite-dimensional case by O. Biquard and P. Gauduchon in [4] (see Theorem 3.1 there), and based on an idea in [14]. In this Subsection, we will again make use of the setting of Subsection 2.2 and we will suppose that the complex structure I of $\mathcal{M}$ is formally integrable.

Recall that the complex structure I acts on n-differential forms by $\eta\mapsto I\eta$ where

$$(I\eta)_x(X_1,\ldots,X_n)\ := (-1)^n\eta(IX_1,\ldots,IX_n).$$

and where x belongs to $\mathcal{M}$, and $X_1,\ldots,X_n$ are elements in $T_x\mathcal{M}$. This action allows one to define the corresponding differential operator d^c by $d^c := IdI^{-1}$. Note that $dd^c = 2i\partial\bar{\partial}$ where ∂ and $\bar{\partial}$ are the Dolbeault operators. Recall also the following definitions :

Definition 2.26 A Kähler potential on $\mathcal{M}$ is a function K on $\mathcal{M}$ such that $\omega = dd^cK$.

In the following we will make the assumption that $\mathcal{M}$ admits a G-invariant globally defined Kähler potential, which will be the case in the next Section. Under this assumption, the action of G is Hamiltonian with respect to the moment map μ defined by:

$$\mu(x)(\mathfrak{a}) \ := dK_x(IX^{\mathfrak{a}}),$$

for all x in $\mathcal{M}$, and for all $\mathfrak{a}$ in $\mathfrak{g}$.

Lemma 2.27 *If K is a globally defined Kähler potential on $(\mathcal{M},\omega,I)$, then the trivial bundle $L=\mathcal{M}\times\mathbb{C}$ endowed with the Chern connection ∇ associated with the Hermitian product h on L given by :*

$$h(\sigma(x),\sigma(x)) \ := e^{-2K(x)},$$

where σ is the canonical section $\sigma(x)=(x,1)$, prequantifies $\mathcal{M}$ in the sense that $R^{\nabla}=i\omega$.

$\triangle$ **Proof of Lemma** 2.27:
Given a non-vanishing section σ such that $\bar{\partial}\sigma=0$, the curvature of the Chern connection has the following expression

$$R^{\nabla}=\frac{1}{2i}dd^c\log h(\sigma,\sigma)$$

since the $\bar{\partial}$ operator on $L=\mathcal{M}\times\mathbb{C}$ satisfies the formal integrability condition $d^{\bar{\partial}}\circ d^{\bar{\partial}}=0$. Hence the Hermitian product h satisfies : $dd^c\log h(\sigma,\sigma)=-2\omega=-2dd^cK$, i.e. $\log h=-2K+\eta$, where η is in the kernel of dd^c and can be chosen to be 0. $\triangle$

The aim of this Subsection is to compute a Kähler potential on a smooth Kähler quotient $\mathcal{M}//G:=\mu^{-1}(\xi)/G$, given a globally defined Kähler potential on $\mathcal{M}$. For this purpose we will construct a holomorphic line bundle $(\hat{L},\hat{h})$ over $\mu^{-1}(\xi)/G$ which prequantifies $\mu^{-1}(\xi)/G$. The previous Lemma establishes the link between the Hermitian scalar product on a trivial line bundle and a globally defined potential. If the pull-back of $(\hat{L},\hat{h})$ to the stable manifold $\mathcal{M}^s$ (which is an open subset of $\mathcal{M}$ since we assume hypothesis (H) satisfied) is a trivial Hermitian line bundle, then the pull-back of the Hermitian scalar product $\hat{h}$ induces a globally defined Kähler potential on $\mathcal{M}^s$.

Consider the following action of the Lie algebra $\mathfrak{g}$ on the vector space $\Gamma(L)$ of sections of the trivial line bundle L :

$$\mathfrak{a}.\sigma=-\nabla_{X^{\mathfrak{a}}}\sigma-i\mu^{\mathfrak{a}}\sigma+i\xi(\mathfrak{a}).\sigma,$$

for all σ in $\Gamma(L)$ and $\mathfrak{a}$ in $\mathfrak{g}$. This action corresponds to the action of $\mathfrak{g}$ on the total space of L which assigns to an element $\mathfrak{a}\in\mathfrak{g}$ the vector field $\hat{X}^{\mathfrak{a}}$ over L whose value at a point $\zeta\in L$ over $x\in\mathcal{M}$ is :

$$\hat{X}^{\mathfrak{a}}(\zeta)=\tilde{X}^{\mathfrak{a}}(\zeta)+i\mu^{\mathfrak{a}}(x).T(\zeta)-i\xi(\mathfrak{a}).T(\zeta),$$

where $\tilde{X}^{\mathfrak{a}}(\zeta)$ denotes the horizontal lift at ζ of $X^{\mathfrak{a}}$ for the trivial connection, and where T denotes the vertical vector field given by $T(\zeta) = \zeta$. This action integrates into an action of the group G if and only if the following integrability condition is satisfied :

(I) $\mathfrak{a} \mapsto i\xi(\mathfrak{a})$ is the differential of a group homomorphism from G to S^1 at the unit element e of G.

Suppose it is the case and denote by χ the homomorphism such that $(d\chi)_e = -i\xi$. The homomorphism χ extends to a homomorphism of $G^{\mathbb{C}}$ to $\mathbb{C}^*$, which will be also denoted by χ. The corresponding action of $G^{\mathbb{C}}$ on L is given by :

$$g \cdot (x\,,\, z_x) \;=\; (g \cdot x\,,\, \chi(g)^{-1}.z_x),$$

where g is an element in $G^{\mathbb{C}}$. This induces an action of $G^{\mathbb{C}}$ on $\Gamma(L)$ by :

$$(g \cdot \sigma)(x) \;:= g(\sigma(g^{-1} \cdot x)).$$

where σ belongs to $\Gamma(L)$, g is an element in $G^{\mathbb{C}}$ and x is in $\mathcal{M}$. In the remainder of this Subsection, we will assume that the integrability condition (I) is satisfied.

Definition 2.28 Let $\hat{L}$ be the complex line bundle over $\mu^{-1}(\xi)/G$ obtained as the G-orbit space of the restriction of the trivial line bundle L to the level set $\mu^{-1}(\xi)$. The fiber of $\hat{L}$ over an element $[x]$ in $\mu^{-1}(\xi)/G$ is :

$$\hat{L}([x]) = [(x\,,\, z_x)],$$

where $(x\,,\, z_x) \sim (g \cdot x\,,\, \chi(g)^{-1}.z_x)$ for all $g \in G^{\mathbb{C}}$.

Definition 2.29 Define an Hermitian scalar product $\hat{h}$ on $\hat{L}$ by :

$$\hat{h}(\hat{\sigma}_1, \hat{\sigma}_2) = h(\sigma_1, \sigma_2),$$

where, for $i \in \{1,\, 2\}$, σ_i is the G-invariant section of $L_{|\mu^{-1}(\xi)}$ whose projection to $\Gamma(\hat{L})$ is $\hat{\sigma}_i$.

Proposition 2.30 *Let σ be a G-invariant section of $L_{|\mu^{-1}(\xi)}$ whose projection to $\Gamma(\hat{L})$ will be denoted by $\hat{\sigma}$, and let X be a vector field on $\mu^{-1}(\xi)/G$ whose horizontal lift with respect to an arbitrary G-invariant connection $\tilde{\nabla}$ on the bundle $\mu^{-1}(\xi) \to \mu^{-1}(\xi)/G$ will be denoted by $\tilde{X}$. Then the identity*

$$\hat{\nabla}_X \hat{\sigma} \;:= \nabla_{\tilde{X}} \sigma$$

where ∇ denotes the Chern connection on L, defines a connection $\hat{\nabla}$ on $\hat{L}$, which is independent of $\tilde{\nabla}$ and for which $\hat{h}$ is parallel.

$\square$ **Proof of Proposition** 2.30 :
Note that : $\hat{\nabla}_X \hat{h} = \nabla_{\tilde{X}} h = 0$, since ∇ preserves h. Let $\tilde{X}_1$ be the horizontal lift of X with respect to another G-invariant connection. One has : $\tilde{X}_1 = \tilde{X} + X^{\mathfrak{a}}$, with $\mathfrak{a} \in \mathfrak{g}$. If σ is G-invariant, then :

$$\nabla_{X^{\mathfrak{a}}} \sigma = -\mathfrak{a}.\sigma - i\mu^{\mathfrak{a}}\sigma + i\xi(\mathfrak{a}).\sigma = 0.$$

Consequently $\nabla_{\tilde{X}}\sigma = \nabla_{\tilde{X}_1}\sigma$, hence $\hat{\nabla}$ is independent of the connection $\tilde{\nabla}$. To check that $\nabla_{\tilde{X}}\sigma$ is G-invariant, note that for all $\mathfrak{a} \in \mathfrak{g}$, one has :

$$\begin{aligned} \mathfrak{a}.\nabla_{\tilde{X}}\sigma &= -\nabla_{X^{\mathfrak{a}}}\nabla_{\tilde{X}}\sigma - i\mu^{\mathfrak{a}}\nabla_{\tilde{X}}\sigma + i\xi(\mathfrak{a})\nabla_{\tilde{X}}\sigma \\ &= -\nabla_{\tilde{X}}\nabla_{X^{\mathfrak{a}}}\sigma - \nabla_{[X^{\mathfrak{a}},\tilde{X}]}\sigma - R_{X^{\mathfrak{a}},\tilde{X}}\sigma \\ &= -i\omega(X^{\mathfrak{a}},\tilde{X})\sigma = 0. \end{aligned}$$

$\square$

Proposition 2.31 *The Hermitian line bundle $(\hat{L}, \hat{h}, \hat{\nabla})$ prequantifies the quotient $\mu^{-1}(\xi)/G$ in the sense that $R^{\hat{\nabla}} = i\omega_{red}$.*

$\square$ **Proof of Proposition** 2.31 :
For any vector fields X and Y on $\mu^{-1}(\xi)/G$, and for every section $\hat{\sigma}$ of $\hat{L}$ given by a G-invariant section σ of $L_{|\mu^{-1}(\xi)}$, one has :

$$\begin{aligned} R^{\hat{\nabla}}_{X,Y}\hat{\sigma} &= \nabla_{\tilde{X}}\nabla_{\tilde{Y}}\sigma - \nabla_{\tilde{Y}}\nabla_{\tilde{X}}\sigma - \nabla_{\widetilde{[X,Y]}}\sigma \\ &= i\omega(\tilde{X},\tilde{Y})\sigma + \nabla_{[\tilde{X},\tilde{Y}]-\widetilde{[X,Y]}}\sigma \\ &= i\omega_{red}(X,Y)\sigma, \end{aligned}$$

since $[\tilde{X},\tilde{Y}] - \widetilde{[X,Y]}$ is tangent to the G-orbit and since, for every $\mathfrak{a} \in \mathfrak{g}$, the identity

$$\mathfrak{a}.\sigma = -\nabla_{X^{\mathfrak{a}}}\sigma - i\mu^{\mathfrak{a}}\sigma + i\xi(\mathfrak{a}).\sigma = 0$$

implies that the covariant derivative of σ with respect to a vertical vector vanishes. $\square$

Corollary 2.32 *The complex structure of the Hermitian line bundle $(\hat{L}, \hat{h})$ is formally integrable and the connection $\hat{\nabla}$ is the Chern connection.*

$\square$ **Proof of Corollary** 2.32 :
The curvature of $\hat{L}$ being of type $(1,1)$, the operator $\bar{\partial} := \hat{\nabla}^{0.1}$ defines a formally integrable complex structure on $\hat{L}$. Moreover, $\hat{\nabla}$ is $\mathbb{C}$-linear and preserves $\hat{h}$, thus it is the associated Chern connection. $\square$

To proceed, suppose that we are under the assumptions of Theorem 2.21, so that we have a submersion:

$$p \ : \ \mathcal{M}^s \ \to \ \mu^{-1}(\xi)/G$$

given by the identification of $\mu^{-1}(\xi)/G$ with $\mathcal{M}^s/G^{\mathbb{C}}$. Note that the differential of p satisfies : $dp \circ I = I_{res} \circ dp$, where I and I_{red} are the complex structures on $\mathcal{M}^s$ and $\mu^{-1}(\xi)/G$ respectively. From Proposition 2.19 it follows that there exists a map

$$g \ : \ \mathcal{M}^s \ \to \exp i\mathfrak{g}$$

and a smooth projection :

$$q \ : \ \mathcal{M}^s \ \to \mu^{-1}(\xi),$$

satisfying $g(x)\cdot x = q(x)$ for all x in $\mathcal{M}^s$.

Proposition 2.33 *The 2-form $p^*\omega_{red}$ is the curvature of the line bundle $(L_{|\mathcal{M}^s}, \bar{h})$, where $\bar{h}$ is defined by:*

$$\bar{h}(\zeta,\zeta) = h\,(\,g(x)\cdot\zeta\,,\,g(x)\cdot\zeta)\,.$$

for x in $\mathcal{M}^s$ and ζ in L_x.

□ **Proof of Proposition** 2.33 :
Since dp satisfies $dp \circ I = I_{res} \circ dp$, the pull-back by p of the Chern connection associated with $\hat{h}$ is the Chern connection of $p^*\hat{L}$ with respect to $p^*\hat{h}$. Thus $R^{p^*\hat{\nabla}} = ip^*\omega_{red}$. In addition, the fiber over $x \in \mathcal{M}^s$ of the bundle $p^*\hat{L}$ is :

$$(p^*\hat{L})_x = \hat{L}_{p(x)} = [L_{q(x)}]_{[q(x)]}.$$

Since the element $q(x)$ in the class $[q(x)]$ is distinguished, the fiber $(x, [L_{q(x)}])$ can be identified with the fiber $(x, L_{q(x)})$. One can therefore define an isomorphism Φ of complex bundles by :

$$\begin{array}{rcl} \Phi \ : \ L_{|\mathcal{M}^s} & \longrightarrow & p^*\hat{L} \\ \zeta \in L_x & \longmapsto & g(x)\cdot\zeta \in \left(p^*\hat{L}\right)_x . \end{array}$$

Clearly $\Phi^*\hat{h} = \bar{h}$. Now let $\bar{\nabla}$ be the Chern connection of the trivial Hermitian bundle $(L_{|\mathcal{M}^s}, \bar{h})$. Its curvature is :

$$R^{\bar{\nabla}} = \Phi^{-1} \circ R^{p^*\hat{\nabla}} \circ \Phi.$$

Since the bundle is a complex line bundle, one has :

$$R^{\bar{\nabla}} = R^{p^*\hat{\nabla}} = ip^*\omega_{red}.$$

□

Theorem 2.34 *The 2-form $p^*\omega_{red}$ on $\mathcal{M}^s$ satisfies $ip^*\omega_{red} = dd^c\hat{K}$, where for every $x \in \mathcal{M}^s$,*

$$\hat{K}(x) \;:=\; K(g(x)\cdot x) \;+\; \frac{1}{2}\log\;|\,\chi\,(g(x))\,|^2.$$

■ **Proof of Theorem** 2.34 :
The curvature of the Chern connection of the Hermitian line bundle $(L_{|\mathcal{M}^s}, \bar{h})$ is given by:

$$R = \frac{1}{2i}dd^c\log\;\bar{h}\,(\sigma,\sigma)\,,$$

where σ is the canonical section. Moreover:

$$\bar{h}(\sigma,\sigma) = h\,(g(x)\cdot\sigma\,,\,g(x)\cdot\sigma) \;=\; |\,\chi(g(x))\,|^{-2}h(\sigma,\sigma).$$

Thus:

$$p^*\omega_{red}(x) \;=\; -\frac{1}{2}dd^c\log\;|\,\chi(g(x))\,|^{-2}h(\sigma,\sigma) = dd^cK(\,g(x)\cdot x\,) + \frac{1}{2}dd^c\log\;|\,\chi(g(x))\,|^2.$$

■

3 An Example of hyperkähler quotient of a Banach manifold by a Banach Lie group

3.1 Notation

Let us summarize the notation used in the remainder of this paper.

H will stand for a separable Hilbert space endowed with an orthogonal decomposition $H = H_+ \oplus H_-$ into two closed infinite-dimensional subspaces. The orthogonal projection from H onto H_+ (resp. H_-) will be denoted by p_+ (resp. p_-). The restriction of $p_\pm$ to a subspace P will generally be denoted by $pr_\pm$ and the space P be specified.

Given two Banach spaces E and F, the set of Fredholm operators from E to F will be denoted by $\mathrm{Fred}(E,F)$, the Hilbert space of Hilbert-Schmidt operators from E to F will be denoted by $L^2(E,F)$, the Banach space of trace class operators from E to F by $L^1(E,F)$, and the Banach space of bounded operators from E to F by $B(E,F)$. The argument F in the previous operator spaces will be omitted when $F = E$.

The set of self-adjoint trace class operators on a complex Hilbert space E will be denoted by $\mathcal{S}^1(E)$, and the set of skew-Hermitian trace class operators on E by $\mathcal{A}^1(E)$. Similarly, $\mathcal{S}^2(E)$ will stand for self-adjoint Hilbert-Schmidt operators on E, and $\mathcal{A}^2(E)$ for skew-Hermitian Hilbert-Schmidt operators on E.

The unitary group of H will be denoted by $\mathcal{U}(H)$ and the identity map of a Hilbert space E by Id_E. $\mathrm{Ran}(A)$ will stand for the range of an operator A and $\mathrm{Ker}(A)$ for its kernel.

3.2 Introduction

The restricted Grassmannian Gr_{res} of H, studied for instance in [38] and [50], is defined as follows:

$$Gr_{res}(H) = \{ \ P \text{ closed subspace of } H \text{ such that } \begin{array}{lcl} pr_+ \ : P & \to & H_+ \in \mathrm{Fred}(P, H_+) \\ pr_- \ : P & \to & H_- \in L^2(P, H_-) \ \}, \end{array}$$

where $pr_\pm$ denotes the orthogonal projection from P to $H_\pm$. The space Gr_{res} is a Hilbert manifold and a homogeneous space under the restricted unitary group:

$$\mathcal{U}_{res} = \left\{ \begin{pmatrix} U_+ & U_{-+} \\ U_{+-} & U_- \end{pmatrix} \in \mathcal{U}(H) \ \mid \ U_{-+} \in L^2(H_-, H_+), \ U_{+-} \in L^2(H_+, H_-) \right\}.$$

Note that the stabilizer of H_+ is $\mathcal{U}(H_+) \times \mathcal{U}(H_-)$. The space Gr_{res} is a strong Kähler manifold whose Kähler structure is invariant under $\mathcal{U}_{res}$. The expressions of the metric g_{Gr}, the complex structure I_{Gr} and the symplectic form ω_{Gr} at the tangent space of Gr_{res} at H_+ are :

$$\begin{array}{l} \mathrm{g}_{Gr}(X, Y) = \Re \ \mathrm{Tr} \ X^*Y \\ I_{Gr}Y = iY \\ \omega_{Gr}(X, Y) = \mathrm{g}_{Gr}(iX, Y) = \Im \ \mathrm{Tr} \ X^*Y, \end{array}$$

where X and Y belong to the tangent space $T_{H_+}Gr_{res}$ which can be identified with $L^2(H_+, H_-)$. Two elements P_1 and P_2 of Gr_{res} are in the same connected component Gr^j_{res}, $(j \in \mathbb{Z})$, if and only if the projections $pr^1_+ \ : P_1 \to H_+$ and $pr^2_+ \ : P_2 \to H_+$ have the same index j. In particular Gr^0_{res} denotes the connected component of Gr_{res} containing H_+.

The aim of this Section is to construct a hyperkähler quotient whose quotient space will be identified with the cotangent space of Gr^0_{res} in Section 4. This will make $T'Gr^0_{res}$ into a strong hyperkähler manifold. In Section 5, the same quotient space will be identified with a complexified orbit of Gr^0_{res}, which will therefore carry a hyperkähler structure in its own right. Since all the constructions that follow can be carried out substituting an arbitrary element of another connected component Gr^j_{res} for H_+ and its orthogonal for H_-, it will follow in particular that the cotangent space of the whole restricted Grassmannian is strongly hyperkähler and that it can be identified with the union of the complexifications of all connected components Gr_{res}^j. The latter union is nothing but the orbit of p_+ under the action by conjugation of the (non-connected) group GL_{res} which is the complexification of $\mathcal{U}_{res}$. In other words, $T'Gr_{res} \simeq GL_{res}{\cdot}p_+$.

3.3 A weak hyperkähler affine space $T\mathcal{M}_k$

For $k \in \mathbb{R}^*$, let $\mathcal{M}_k$ be the following affine Banach space :

$$\mathcal{M}_k \;:= \left\{ \; x = \left(\begin{array}{c} x_+ \\ x_- \end{array} \right) \in B(H_+, H) \;\;|\;\; x_+ - \;\mathrm{Id}_{H_+} \in L^1(H_+),\; x_- \in L^2(H_+, H_-) \right\}.$$

Then $\mathcal{M}_k$ is modelled over the Banach space $L^1(H_+, H_+) {\times} L^2(H_+, H_-)$.
The tangent space of $\mathcal{M}_k$

$$T\mathcal{M}_k = \left\{ \; (x, X) \in \; \mathcal{M}_k \times B(H_+, H) \;\;|\;\; p_+ \circ X \in L^1(H_+),\; p_- \circ X \in L^2(H_+, H_-) \; \right\},$$

injects into the continuous cotangent space $T'\mathcal{M}_k$ of $\mathcal{M}_k$ via the application :

$$(x, X) \mapsto (x, (Y \mapsto \;\; \mathrm{Tr}\; X^*Y)).$$

Thus $T\mathcal{M}_k$ inherits a structure of weak complex symplectic manifold with symplectic form Ω given by

$$\Omega\left((Z_1, T_1)\;;(Z_2, T_2)\right) = \;\mathrm{Tr}\;(T_1^* Z_2) - \;\mathrm{Tr}\;(T_2^* Z_1),$$

where (Z_1, T_1) and (Z_2, T_2) belong to $T_{(x,X)}(T\mathcal{M}_k)$. We will denote by ω_2 and ω_3 the real symplectic forms given respectively by the real and imaginary parts of Ω. Besides, from the natural inclusion of $L^1(H_+)$ into $L^2(H_+)$, it follows that $T\mathcal{M}_k$ admits a natural weak Riemannian metric whose expression is

$$\mathrm{g}_{(x,X)}\left((Z_1, T_1)\;;(Z_2, T_2)\right) = \Re\;\mathrm{Tr}\; Z_1^* Z_2 + \Re\;\mathrm{Tr}\; T_1^* T_2$$

where (Z_1, T_1) and (Z_2, T_2) belong to $T_{(x,X)}(T\mathcal{M}_k)$. The complex symplectic form Ω and the Riemannian metric g give rise to a hyperkähler structure on $T\mathcal{M}_k$ with complex structures :

$$\begin{array}{ll} I_1(Z, T) & = (iZ, -iT) \\ I_2(Z, T) & = (T, -Z) \\ I_3(Z, T) & = (iT, iZ). \end{array}$$

The real symplectic form associated with I_1 is given by

$$\begin{array}{ll} \omega_1\left((Z_1, T_1)\;;(Z_2, T_2)\right) & = \mathrm{g}_{(x,X)}\left((I_1(Z_1, T_1)\;;(Z_2, T_2)\right) \\ & = \Im\;\mathrm{Tr}\; Z_1^* Z_2 - \Im\;\mathrm{Tr}\; T_1^* T_2. \end{array}$$

3.4 Tri-Hamiltonian action of a unitary group G

Let G be the following Banach Lie group of unitary operators :

$$G \ := \mathcal{U}(H_+) \ \cap \ \{ \ \mathrm{Id} + L^1(H_+) \ \}.$$

The Lie algebra $\mathfrak{g}$ of G is the Lie algebra of skew-Hermitian operators of trace class. We will denote by $G^{\mathbb{C}}$ the complexification of G :

$$G^{\mathbb{C}} \ := GL(H_+) \ \cap \ \{ \ \mathrm{Id} + L^1(H_+) \ \},$$

and by $\mathfrak{g}^{\mathbb{C}}$ its complex Lie algebra $\mathfrak{g} \oplus i\mathfrak{g}$. The group G acts on $T\mathcal{M}_k$ by

$$u \cdot (x, X) := (x \circ u^{-1}, \, X \circ u^{-1}),$$

for all u in G and for all (x, X) in $T\mathcal{M}_k$. This action extends to an I_1-holomorphic action of $G^{\mathbb{C}}$ by

$$g \cdot (x, X) := (x \circ g^{-1}, \, X \circ g^{*}),$$

for all g in $G^{\mathbb{C}}$ and for all (x, X) in $T\mathcal{M}_k$.

Proposition 3.1 *The action of $G^{\mathbb{C}}$ on $T\mathcal{M}_k$ is Hamiltonian with respect to the complex symplectic form Ω, with moment map $\mu^{\mathbb{C}}$:*

$$\begin{array}{cccl} \mu^{\mathbb{C}} & : & T\mathcal{M}_k & \longrightarrow & (\mathfrak{g}^{\mathbb{C}})' \\ & & (x, X) & \longmapsto & (\mathfrak{a} \mapsto \ \mathit{Tr} \ (X^* x \mathfrak{a}))\,. \end{array}$$

□ **Proof of Proposition** 3.1:
Let us check that $\mu^{\mathbb{C}}$ satisfies:

$$\langle d\mu^{\mathbb{C}}_{(x,X)} \left((Z, T) \right), \mathfrak{a} \rangle = i_{\mathfrak{a} \cdot (x,X)} \Omega \left((Z, T) \right),$$

for all (Z, T) in $T_{(x,X)}\mathcal{M}_k$ and for all $\mathfrak{a}$ in $\mathfrak{g}^{\mathbb{C}}$, where $\langle \ , \ \rangle$ denotes the duality pairing and where $\mathfrak{a} \cdot (x, X) = (-x \circ \mathfrak{a}, \, X \circ \mathfrak{a}^*)$ is the vector induced by the infinitesimal action of $\mathfrak{a}$ on (x, X). One has :

$$\begin{array}{rcll} \langle d\mu^{\mathbb{C}}_{(x,X)} \left((Z, T) \right), \mathfrak{a} \rangle & = & \mathrm{Tr} \ \left((X^* Z + T^* x) \mathfrak{a} \right) = \ \mathrm{Tr} \ (X^* Z \mathfrak{a} + T^* x \mathfrak{a}) & \\ & = & \mathrm{Tr} \ (\mathfrak{a} X^* Z) - \ \mathrm{Tr} \ (T^* (-x \circ \mathfrak{a})) & \text{since } \mathfrak{a} \in L^1(H_+) \\ & = & \mathrm{Tr} \ \left((X \circ \mathfrak{a}^*)^* Z \right) - \ \mathrm{Tr} \ (T^* (-x \circ \mathfrak{a})) & \\ & = & i_{(-x \circ \mathfrak{a}, X \circ \mathfrak{a}^*)} \Omega \left((Z, T) \right). & \end{array}$$

□

It follows from Proposition 3.1 that the real symplectic forms ω_2 and ω_3 are Hamiltonian with respect to the real moment maps :

$$\begin{array}{rccl} \mu_2 = \Re(\mu^{\mathbb{C}}) : & T\mathcal{M}_k & \rightarrow & \mathfrak{g}' \\ & (x, X) & \mapsto & \left(\mathfrak{a} \mapsto \frac{1}{2} \ \mathrm{Tr} \ \left(X^* x - x^* X \right) \mathfrak{a} \right) \end{array}$$

$$\begin{array}{rccl} \mu_3 = \Im(\mu^{\mathbb{C}}) : & T\mathcal{M}_k & \rightarrow & \mathfrak{g}' \\ & (x, X) & \mapsto & \left(\mathfrak{a} \mapsto -\frac{i}{2} \ \mathrm{Tr} \ \left(X^* x + x^* X \right) \mathfrak{a} \right) \end{array}$$

Proposition 3.2 *The action of G on $T\mathcal{M}_k$ is Hamiltonian with respect to the real symplectic form ω_1 with moment map :*

$$\begin{array}{rccl} \mu_1 & : & T\mathcal{M}_k & \longrightarrow & \mathfrak{g}' \\ & & (x,X) & \longmapsto & \left(\mathfrak{a} \mapsto -\frac{i}{2}\ \mathit{Tr}\ (x^*x - X^*X)\,\mathfrak{a}\right). \end{array}$$

$\square$ **Proof of Proposition** 3.2 :
One has to check that μ_1 satisfies :

$$\langle (d\mu_1)_{(x,X)}\left((Z,T)\right), \mathfrak{a}\rangle = i_{(-x\circ\mathfrak{a}, -X\circ\mathfrak{a})}\omega_1\left((Z,T)\right),$$

for all (Z,T) in $T_{(x,X)}\mathcal{M}_k$ and all $\mathfrak{a}$ in $\mathfrak{g}$. One has :

$$\begin{array}{rl} \langle (d\mu_1)_{(x,X)}\left((Z,T)\right), \mathfrak{a}\rangle & = -\frac{i}{2}\ \mathrm{Tr}\ \left(Z^*x\mathfrak{a} + x^*Z\mathfrak{a}\right) + \frac{i}{2}\ \mathrm{Tr}\ \left(T^*X\mathfrak{a} + X^*T\mathfrak{a}\right) \\ & = \frac{i}{2}\ \mathrm{Tr}\ \left(Z^*(-x\circ\mathfrak{a}) - \mathfrak{a}x^*Z\right) - \frac{i}{2}\ \mathrm{Tr}\ \left(T^*(-X\circ\mathfrak{a}) - \mathfrak{a}X^*T\right) \\ & = \frac{i}{2}\ \mathrm{Tr}\ \left(Z^*(-x\circ\mathfrak{a}) - (x\circ\mathfrak{a}^*)^*Z\right) - \frac{i}{2}\ \mathrm{Tr}\ \left(T^*(-X\circ\mathfrak{a}) - (X\circ\mathfrak{a}^*)^*T\right) \\ & = \frac{i}{2}\ \mathrm{Tr}\ \left(Z^*(-x\circ\mathfrak{a}) - (-x\circ\mathfrak{a})^*Z\right) - \frac{i}{2}\ \mathrm{Tr}\ \left(T^*(-X\circ\mathfrak{a}) - (-X\circ\mathfrak{a})^*T\right) \\ & = \Im\ \mathrm{Tr}\ (-x\circ\mathfrak{a})^*Z - \Im\ \mathrm{Tr}\ (-X\circ\mathfrak{a})^*T \\ & = i_{(-x\circ\mathfrak{a}, -X\circ\mathfrak{a})}\omega_1\left((Z,T)\right). \end{array}$$

$\square$

In the following, we will denote by μ the $\mathfrak{g}'\otimes\mathbb{R}^3$-valued moment map defined by:

$$\begin{array}{rccl} \mu & : & T\mathcal{M}_k & \rightarrow & \mathfrak{g}'\otimes\mathbb{R}^3 \\ & & (x,X) & \mapsto & (\,\mu_1(x,X)\ ,\ \mu_2(x,X)\ ,\ \mu_3(x,X)\,). \end{array}$$

In the next Subsection we will consider the $\mathrm{Ad}^*(G)$-invariant value $\xi_k := \left(-\frac{i}{2}k^2\ \mathrm{Tr}\ ,\ 0,\ 0\right)$ of the moment map μ and the level set:

$$\mathcal{W}_k := \mu^{-1}\left(\xi_k\right).$$

3.5 Smooth Banach manifold structure on the level set $\mathcal{W}_k$

This Subsection is devoted to the proof of the following Theorem :

Theorem 3.3 $\mathcal{W}_k := \mu^{-1}\left(\left(-\frac{i}{2}k^2\ \mathit{Tr}\ ,\ 0,\ 0\right)\right)$ *is a smooth Riemannian submanifold of $T\mathcal{M}_k$.*

We will prove that $\xi_k = \left(-\frac{i}{2}k^2\ \mathrm{Tr}\ ,\ 0,\ 0\right)$ is a regular value of the moment map μ. For this purpose we will need the following fact, which will be useful in other parts of the paper, so that we single it out here :

Lemma 3.4 *Let B be a Banach space which injects continuously into a Hilbert space H. Let F be a closed subspace of B and $\bar{F}$ its closure in H. If the orthogonal projection of H onto $\bar{F}$ maps B onto F, then F admits a closed complement in B which is $\bar{F}^{\perp}\ \cap\ B$.*

$\triangle$ **Proof of Lemma** 3.4:
Since the orthogonal projection p of H onto $\bar{F}$ maps B onto F, B is the algebraic sum of F and $\bar{F}^{\perp} \cap B$. Since B injects continuously into H and p is continuous, the projection from B onto F with respect to $\bar{F}^{\perp} \cap B$ is continuous. Hence

$$B = F \oplus (\bar{F}^{\perp} \cap B)$$

as a topological sum. $\triangle$

■ **Proof of Theorem** 3.3:
Consider the following smooth map of Banach manifolds

$$\begin{array}{rcl} \mathcal{F} : T\mathcal{M}_k & \longrightarrow & L^1(H_+) \times \mathcal{S}^1(H_+) \\ (x, X) & \longmapsto & (X^*x,\ x^*x - X^*X), \end{array}$$

We have:

$$\begin{array}{rcl} d_{(x,X)}\mathcal{F} : T_{(x,X)}T\mathcal{M}_k & \longrightarrow & L^1(H_+) \times \mathcal{S}_1(H_+) \\ (Z, T) & \longmapsto & (X^*Z + T^*x,\ x^*Z + Z^*x - X^*T - T^*X). \end{array}$$

The level set $\mathcal{W}_k$ is the preimage of $(0, k^2)$ under $\mathcal{F}$. To prove that $\mathcal{W}_k$ is a smooth Banach submanifold of $T\mathcal{M}_k$ it is sufficient to prove that the differential of $\mathcal{F}$ at a point (x, X) of $\mathcal{W}_k$ is surjective and that its kernel splits.

. For this purpose consider the following decomposition of H_+:

$$H_+ = \text{ Ker } X \oplus (\text{ Ker } X)^{\perp}. \tag{4}$$

The operator x is a Fredholm operator hence it has closed range, and X is a compact operator. The equality $X^*x = 0$ implies that the range $\operatorname{Ran} x$ of x is orthogonal to the range $\operatorname{Ran} X$ of X. From the continuity of the orthogonal projection of H onto $\operatorname{Ran} x$ it follows that: $\operatorname{Ran} x \perp \overline{\operatorname{Ran} X}$. Let us introduce the following decomposition of H:

$$H = \overline{\text{Ran } X} \oplus (\text{ Ran } X)^{\perp} \cap (\text{ Ran } x)^{\perp} \oplus \text{ Ran } x_{|\text{ Ker } X} \oplus \text{ Ran } x_{|\text{ Ker } X^{\perp}}. \tag{5}$$

With respect to the decompositions (4) and (5) of H_+ and H into closed subspaces, x and X have the following expressions :

$$x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x_{31} & 0 \\ 0 & x_{42} \end{pmatrix} \quad X = \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where x_{31} and x_{42} are continuous bijections, thus isomorphisms, and where X_{12} is 1-1 but not onto. Let (Z, T) be a tangent vector to

$T\mathcal{M}_k$ at (x,X), and denote by $(Z_{ij})_{1\leq i\leq 4,\,1\leq j\leq 2}$ and $(T_{ij})_{1\leq i\leq 4,\,1\leq j\leq 2}$ the block decompositions of Z and T with respect to the direct sums (4) and (5). Note that :

$$X^*Z + T^*x = \begin{pmatrix} T_{31}^*x_{31} & T_{41}^*x_{42} \\ X_{12}^*Z_{11} + T_{32}^*x_{31} & X_{12}^*Z_{12} + T_{42}^*x_{42} \end{pmatrix},$$

and that $x^*Z + Z^*x - X^*T - T^*X$ equals :

$$\begin{pmatrix} x_{31}^*Z_{31} + Z_{31}^*x_{31} & x_{31}^*Z_{32} + Z_{41}^*x_{42} - T_{11}^*X_{12} \\ x_{42}^*Z_{41} + Z_{32}^*x_{31} - X_{12}^*T_{11} & x_{42}^*Z_{42} + Z_{42}^*x_{42} - T_{12}^*X_{12} - X_{12}^*T_{12} \end{pmatrix}.$$

. To show that the differential of $\mathcal{F}$ is onto, consider an element (U,V) of $L^1(H_+)\times\mathcal{S}^1(H_+)$ and denote be $(U_{ij})_{1\leq i,j\leq 2}$ and $(V_{ij})_{1\leq i,j\leq 2}$ the block decompositions of U and V with respect to the direct sum $H_+ = \text{ Ker } X \oplus (\text{ Ker } X)^\perp$. A preimage of (U,V) by $d_{(x,X)}\mathcal{F}$ is given by the following ordered pair (Z,T):

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{2}x_{31}^{-1*}V_{11} & \frac{1}{2}x_{31}^{-1*}V_{12} \\ \frac{1}{2}x_{42}^{-1*}V_{21} & \frac{1}{2}x_{42}^{-1*}V_{22} \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x_{31}^{-1*}U_{11}^* & x_{31}^{-1*}U_{21}^* \\ x_{42}^{-1*}U_{12}^* & x_{42}^{-1*}U_{22}^* \end{pmatrix}.$$

. Let us now show that for every (x,X) in $\mathcal{W}_k$, the kernel of the differential $d_{(x,X)}\mathcal{F}$ splits. It is given by the following subspace of $T_{(x,X)}\mathcal{M}_k$:

$$T_{(x,X)}\mathcal{W}_k = \{\,(Z,T) \in T_{(x,X)}T\mathcal{M}_k \mid X^*Z+T^*x = 0,\ x^*Z+Z^*x = X^*T+T^*X\,\}.$$

Consider the space $L^2(H_+,H)\times L^2(H_+,H)$ endowed with the complex structure $I(Z,T) = (iZ,-iT)$ and the strong Riemannian metric $\bar{\mathrm{g}}$ given by the real part of the natural Hermitian product. Let E be the closure of $T_{(x,X)}\mathcal{W}_k$ in $L^2(H_+,H)\times L^2(H_+,H)$ and $E^{\perp_{\bar{\mathrm{g}}}}$ its orthogonal in $L^2(H_+,H)\times L^2(H_+,H)$. By Lemma 3.4, to show that $T_{(x,X)}\mathcal{W}_k$ is complemented in $T_{(x,X)}(T\mathcal{M}_k)$, it is sufficient to show that the orthogonal projection of $L^2(H_+,H)\times L^2(H_+,H)$ onto E maps $T_{(x,X)}(T\mathcal{M}_k)$ onto $T_{(x,X)}\mathcal{W}_k$. An ordered pair $(Z,T)\in L^2(H_+,H)\times L^2(H_+,H)$ is in E if and only if Z and T are of the following form:

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ x_{31}^{-1*}\mathfrak{a}_1 & Z_{32} \\ x_{42}^{-1*}(X_{12}^*T_{11} - Z_{32}^*x_{31}) & x_{42}^{-1*}(\frac{1}{2}(X_{12}^*T_{12} + T_{12}^*X_{12}) + \mathfrak{a}_2) \end{pmatrix}$$

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ 0 & -x_{31}^{-1*}Z_{11}^*X_{12} \\ 0 & -x_{42}^{-1*}Z_{12}^*X_{12} \end{pmatrix},$$

where $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$) is an element of the space $\mathcal{A}^2(\operatorname{Ker} X)$ (resp. $\mathcal{A}^2\left((\operatorname{Ker} X)^{\perp}\right)$ of skew-Hermitian Hilbert-Schmidt operators on $\operatorname{Ker} X$ (resp. on $(\operatorname{Ker} X)^{\perp}$). An ordered pair $(Z,T) \in L^2(H_+,H)\times L^2(H_+,H)$ is in $E^{\perp_{\bar{\mathrm{g}}}}$ if and only if Z and T have the following form:

$$Z = \begin{pmatrix} X_{12}T_{32}^*x_{31}^{-1*} & X_{12}T_{42}^*x_{42}^{-1*} \\ 0 & 0 \\ x_{31}\mathfrak{s}_1 & x_{31}Z_{41}^*x_{42}^{-1*} \\ Z_{41} & x_{42}\mathfrak{s}_2 \end{pmatrix} \quad T = \begin{pmatrix} -X_{12}x_{42}^{-1}Z_{41} & -X_{12}\mathfrak{s}_2 \\ 0 & 0 \\ T_{31} & T_{32} \\ T_{41} & T_{42} \end{pmatrix},$$

where $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$) is an element of the space $\mathcal{S}^2(\operatorname{Ker} X)$ (resp. $\mathcal{S}^2\left((\operatorname{Ker} X)^{\perp}\right)$ of self-adjoint Hilbert-Schmidt operators on $\operatorname{Ker} X$ (resp. $(\operatorname{Ker} X)^{\perp}$). We will denote by $(p_E(Z),\ p_E(T))$ the orthogonal projection of an element $(Z,T) \in L^2(H_+,H) \times L^2(H_+,H)$ to E, and $(p_E(Z_{ij}))_{1\leq i\leq 4,\,1\leq j\leq 2}$ (resp. $(p_E(T_{ij}))_{1\leq i\leq 4,\,1\leq j\leq 2}$) the block decomposition of $p_E(Z)$ (resp. $p_E(T)$) with respect to the direct sums (4) and (5). Then $p_E(Z)$ and $p_E(T)$ are given by :

$$p_E(Z) = \begin{pmatrix} p_E(Z_{11}) & p_E(Z_{12}) \\ Z_{21} & Z_{22} \\ \frac{1}{2}(Z_{31} - x_{31}^{-1*}Z_{31}^*x_{31}) & p_E(Z_{32}) \\ p_E(Z_{41}) & \frac{1}{2}x_{42}^{-1*}\left(\frac{1}{2}\left(X_{12}^*p_E(T_{12}) + p_E(T_{12})^*X_{12}\right) + \mathfrak{a}_1\right) \end{pmatrix}$$

$$p_E(T) = \begin{pmatrix} p_E(T_{11}) & p_E(T_{12}) \\ T_{21} & T_{22} \\ 0 & -x_{31}^{-1*}(p_E(Z_{11}))^*X_{12} \\ 0 & -x_{42}^{-1*}(p_E(Z_{12}))^*X_{12} \end{pmatrix}.$$

with:

$$\begin{aligned} &p_E(Z_{41}) = \tfrac{1}{2}(Z_{41} + x_{42}^{-1*}Z_{32}^*x_{31} - x_{42}^{-1*}X_{12}^*T_{12}) \\ &p_E(Z_{32}) = Z_{32} - x_{31}p_E(Z_{41})^*x_{42}^{-1*} \\ &p_E(T_{11}) = T_{11} + X_{12}x_{42}^{-1}p_E(Z_{41}) \\ &\mathfrak{a}_1 = \tfrac{1}{2}(x_{42}^*Z_{42} - Z_{42}^*x_{42}) + \tfrac{1}{2}\left(X_{12}^*\left(T_{12} - p_E(T_{12})\right) - \left(T_{12} - p_E(T_{12})\right)^*X_{12}\right). \end{aligned}$$

Denoting by p_+ the orthogonal projection of H onto H_+, let us prove that $p_+\circ p_E(Z)$ and $p_+\circ p_E(T)$ are of trace class whenever $p_+(Z)$ and $p_+(T)$ are of trace class. Let

$$p_+ = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{pmatrix}$$

be the decomposition of p_+ with respect to the decompositions (5) of H and (4) of H_+. For $(x,X) \in \mathcal{W}_k$, the operators p_{11}, p_{12}, p_{14}, p_{21}, p_{22}, p_{23} are of trace class, as well as $p_{13} - x_{31}^{-1}$ and $p_{24} - x_{42}^{-1}$. The condition $p_+(Z) \in L^1(H_+)$ implies that Z_{31}, Z_{32}, Z_{41} and Z_{42} are of trace class.

It follows that $p_E(Z_{32})$ and $p_E(Z_{41})$ are of trace class, as well as $\mathfrak{a}_1$. Hence $p_+ \circ p_E(Z)$ is of trace class. On the other hand, p_{11}, p_{12}, p_{21} and p_{22} being of trace class, the condition $p_+(T) \in L^1(H_+)$ implies that $p_+ \circ p_E(T)$ is of trace class. Thus we obtain the following direct sum of closed subspaces :

$$T_{(x,X)}(T\mathcal{M}_k) = T_{(x,X)}\mathcal{W}_k \oplus \left(E^{\perp_{\mathrm{g}}} \cap T_{(x,X)}T\mathcal{M}_k\right).$$

and the Theorem follows. ■

3.6 Hyperkähler quotient of $T\mathcal{M}_k$ by G

In this Subsection we will prove that the assumptions (D) and (S) of Subsection 2.3 are satisfied, as well as the following Theorem.

Theorem 3.5 *The hyperkähler quotient of the weak hyperkähler space $T\mathcal{M}_k$ with respect to the tri-symplectic action of G provides $\mathcal{W}_k/G$ with a structure of strong hyperkähler smooth Hilbert manifold.*

For the remainder of the paper, we will denote by $(\mathrm{g}_{red}, I_1^{red}, I_2^{red}, I_3^{red}, \omega_1^{red}, \omega_2^{red}, \omega_3^{red})$ the hyperkähler structure induced by $T\mathcal{M}_k$ on the reduced space $\mathcal{W}_k/G$.

■ **Proof of Theorem** 3.5:

. Let us first show that the quotient space $\mathcal{W}_k/G$ endowed with the quotient topology is Hausdorff. Since G acts freely on $\mathcal{W}_k$, by Propositions 2.5 and 2.6, it it sufficient to prove that the graph $\mathcal{C}$ of the equivalence relation defined by G is closed in $\mathcal{W}_k \times \mathcal{W}_k$ and that the canonical application from $\mathcal{C}$ to G is continuous.

Note that, for all (y, Y) in $\mathcal{W}_k$, the operator y is an injective Fredholm operator from H_+ to H, thus establishes an isomorphism between H_+ and its (closed) range. We will denote by y^{-1} the inverse operator of y, and extend it to an operator from H to H by demanding that the restriction of y^{-1} to the orthogonal complement of $\operatorname{Ran} y$ vanishes.

Let us remark that the canonical map from the graph $\mathcal{C}$ to G assigns to an ordered pair $((x, X),\ (y, Y))$ the element $g = y^{-1}x$. It is continuous with respect to the topology of $\mathcal{C}$ inherited from the topology of $\mathcal{W}_k \times \mathcal{W}_k$, and the topology of G induced by the L^1-norm on $\mathrm{Id} + L^1(H_+)$.

Let $\{\ (x_n\,,\, X_n)\ ;\ (x_n \circ g_n^{-1},\ X_n \circ g_n^{-1})\ \}$ be a sequence in $\mathcal{C}$ which converges to an element $((x,\ X);\ (y,\ Y))$ in $\mathcal{W}_k \times \mathcal{W}_k$. The sequence $\{\ g_n^{-1} = x_n^{-1} \circ (x_n \circ g_n^{-1})\ \}$ is a sequence of elements of G which converges to the element $x^{-1}y$ in $\mathrm{Id} + L^1(H_+)$. Since G is closed in $\mathrm{Id} + L^1(H_+)$, it follows that $x^{-1}y$ is in G and that $\mathcal{C}$ is closed in $\mathcal{W}_k \times \mathcal{W}_k$.

. By Theorem 2.22, for $\mathcal{W}_k/G$ to be hyperkähler, it is sufficient to have the following topological direct sum, referred to as (D) :

$$\text{(D)} \qquad T_{(x,\,X)}G\cdot(x,\,X) \;\oplus\; \left(T_{(x,\,X)}G\cdot(x,\,X)\right)^{\perp_{\mathrm{g}}} \;=\; T_{(x,\,X)}\mathcal{W}_k.$$

For this purpose, we will use Lemma 3.4. Let us again consider the space $L^2(H_+, H) \times L^2(H_+, H)$ endowed with the complex structure $I(Z,T) = (iZ, -iT)$ and the strong Riemannian metric $\bar{\mathrm{g}}$ given by the real part of the natural Hermitian product. Let E (resp. F) be the closure of $T_{(x,X)}\mathcal{W}_k$ (resp. $T_{(x,X)}G\cdot(x,X)$) in $L^2(H_+, H) \times L^2(H_+, H)$. We will show that the $\bar{\mathrm{g}}$-orthogonal projection of E onto F maps $T_{(x,X)}\mathcal{W}_k$ onto $T_{(x,X)}G\cdot(x,X)$. The orthogonal of F in E is the set of ordered pairs $(Z,T) \in L^2(H_+, H) \times L^2(H_+, H)$ of the form :

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ 0 & x_{31}^{-1*}T_{11}^*X_{12} \\ 0 & x_{42}^{-1*}T_{12}^*X_{12} \end{pmatrix}; \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ 0 & -x_{31}^{-1*}Z_{11}^*X_{12} \\ 0 & -x_{42}^{*-1}Z_{12}^*X_{12} \end{pmatrix}.$$

The orthogonal projection of E onto F maps an ordered pair $(Z,\,T) \in E$ having the following block decompositions with respect to the direct sums (4) and (5)

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ x_{31}^{-1*}\mathfrak{a}_1 & Z_{32} \\ x_{42}^{-1*}(X_{12}^*T_{11} - Z_{32}^*x_{31}) & x_{42}^{-1*}(\frac{1}{2}(X_{12}^*T_{12} + T_{12}^*X_{12}) + \mathfrak{a}_2) \end{pmatrix}$$

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ 0 & -x_{31}^{-1*}Z_{11}^*X_{12} \\ 0 & -x_{42}^{-1*}Z_{12}^*X_{12} \end{pmatrix}$$

to an ordered pair $(p_G(Z), p_G(T))$ with decompositions

$$p_G(Z) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x_{31}^{*-1}\mathfrak{a}_1 & -x_{31}(T_{11}^*X_{12} - x_{31}^*Z_{32})x_{42}^{-1}x_{42}^{-1*} \\ x_{42}^{-1*}(X_{12}^*T_{11} - Z_{32}^*x_{31}) & -x_{42}\mathfrak{a} \end{pmatrix}$$

$$p_G(T) = \begin{pmatrix} X_{12}x_{42}^{-1}x_{42}^{-1*}(X_{12}^*T_{11} - Z_{32}^*x_{31}) & -X_{12}\mathfrak{a} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathfrak{a}$ satisfies :

$$\mathfrak{a}_2 + \frac{1}{2}(X_{12}^* T_{12} - T_{12}^* X_{12}) = k^2 \mathfrak{a} - (x_{42}^* x_{42} \mathfrak{a} + \mathfrak{a} x_{42}^* x_{42}). \qquad (6)$$

Let us show that the projection p_G from E to F maps an element of $T_{(x,X)}\mathcal{W}_k$ into $T_{(x,X)}G\cdot(x,X)$. Let

$$p_+ = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{pmatrix}$$

be the expression of the orthogonal projection onto H_+ with respect to the direct sums (5) and (4). For $(x,X) \in \mathcal{W}_k$, the operators $p_{11}, p_{12}, p_{14}, p_{21}, p_{22}, p_{23}$ are of trace class as well as $p_{13} - x_{31}^{-1}$ and $p_{24} - x_{42}^{-1}$. The condition $p_+(Z) \in L^1(H_+)$ implies in particular that $\mathfrak{a}_1$, Z_{32} and $\mathfrak{a}_2$ are of trace class. To conclude that p_G maps $T_{(x,X)}\mathcal{W}_k$ into $T_{(x,X)}G(x,X)$, it remains to show that $\mathfrak{a}$ defined by Equation (6) is of trace class. This will follow by lemma 3.6 below. As a consequence, $p_+ \circ p_G(Z)$ and $p_+ \circ p_G(T)$ are of trace class. Hence, $F^{\perp_{\bar{\mathrm{g}}}} \cap T_{(x,X)}\mathcal{W}_k$ is a closed complement to $T_{(x,X)}G\cdot(x,X)$. Since g is the restriction of $\bar{\mathrm{g}}$ to $T_{(x,X)}\mathcal{W}_k$, it follows that :

$$T_{(x,X)}G\cdot(x,X) \oplus \left(T_{(x,X)}G\cdot(x,X)\right)^{\perp_{\mathrm{g}}} = T_{(x,X)}\mathcal{W}_k.$$

■

Lemma 3.6 *The map $\mathcal{E}_{x_{42}}$ defined by*

$$\begin{array}{cccll} \mathcal{E}_{x_{42}} & : & L^2\left((\text{ Ker } X)^\perp\right) & \longrightarrow & L^2((\text{ Ker } X)^\perp) \\ & & \mathfrak{a} & \mapsto & k^2\mathfrak{a} - (x_{42}^* x_{42}\mathfrak{a} + \mathfrak{a} x_{42}^* x_{42}). \end{array}$$

is an isomorphism which restricts to an isomorphism of $L^1\left((\text{ Ker } X)^\perp\right)$.

△ **Proof of Lemma** 3.6 :
Indeed, $x_{42}^* x_{42}$ is a self-adjoint positive definite operator on the space $(\text{ Ker } X)^\perp$ satisfying

$$x_{42}^* x_{42} = k^2 \text{ Id } + X_{12}^* X_{12},$$

where $X_{12}^* X_{12}$ is a compact operator. It follows that there exists a diagonal operator D with respect to an orthogonal basis $\{f_i\}_{i\in J}$ of $(\text{ Ker } X)^\perp$, and a unitary operator $u \in \mathcal{U}\left((\text{ Ker } X)^\perp\right)$ such that $x_{42}^* x_{42} = u\ D\ u^*$. Denote by D_i the eigenvalues of D. Remark that $D_i > k^2$. It follows that the equation

$$k^2\mathfrak{a} = x_{42}^* x_{42}\mathfrak{a} + \mathfrak{a} x_{42}^* x_{42}$$

is equivalent to

$$k^2 u^* \ \mathfrak{a} \ u = u^* \ \mathfrak{a} \ u \ D + D \ u^* \ \mathfrak{a} \ u,$$

and implies in particular that, for every i and j in J, one has :

$$k^2 \langle \ (u^* \ \mathfrak{a} \ u)(f_j), \ f_i \ \rangle = (D_j + D_i) \ \langle \ (u^* \ \mathfrak{a} \ u)(f_j), \ f_i \ \rangle.$$

From $(D_j + D_i) - k^2 > k^2$, we get that $\langle \ (u^* \ \mathfrak{a} \ u)(f_j), \ f_i \ \rangle = 0$ for every $i, \ j \in J$. Thus $\operatorname{Ker} \mathcal{E}_{x_{42}} = 0$. To see that the map $\mathcal{E}_{x_{42}}$ is surjective, consider an element V in $L^2((\operatorname{Ker} X)^\perp)$, and define an operator $\tilde{\mathfrak{a}} \in L^2((\operatorname{Ker} X)^\perp)$ by :

$$\langle \ \tilde{\mathfrak{a}}(f_j), \ f_i \ \rangle = \frac{1}{(k^2 - (D_i + D_j))} \ \langle \ (u^* \ V \ u)(f_j), \ f_i \ \rangle.$$

The operator $u \ \tilde{\mathfrak{a}} \ u^*$ is a preimage of V by $\mathcal{E}_{x_{42}}$. Moreover if $V \in L^1((\operatorname{Ker} X)^\perp)$, then $u \ \tilde{\mathfrak{a}} \ u^* \in L^1((\operatorname{Ker} X)^\perp)$. Since $\mathcal{E}_{x_{42}}$ is clearly continuous, it follows that $\mathcal{E}_{x_{42}}$ is an isomorphism of $L^2((\operatorname{Ker} X)^\perp)$ that restricts to an isomorphism of $L^1((\operatorname{Ker} X)^\perp)$. △

Proposition 3.7 *For every $(x, \ X)$ in $\mathcal{W}_k$ one has :*

$$\begin{aligned} T_{(x,X)}\mathcal{M}_k &= T_{(x,X)}G\cdot(x,X) \oplus H_{(x,X)} \oplus I_1\left(T_{(x,X)}G\cdot(x,X)\right) \\ &\quad \oplus I_2\left(T_{(x,X)}G\cdot(x,X)\right) \oplus I_3\left(T_{(x,X)}G\cdot(x,X)\right), \end{aligned}$$

where $H_{(x,\,X)}$ is the orthogonal of $T_{(x,X)}G\cdot(x,X)$ in $T_{(x,\,X)}\mathcal{W}_k$.

□ **Proof of Proposition** 3.7:
With the previous notation, it follows from the proof of Theorem 3.3 that the orthogonal projection from $L^2(H_+, H) \times L^2(H_+, H)$ to $\overline{T_{(x,X)}\mathcal{W}_k}$ with respect to the strong Riemannian metric $\bar{\mathrm{g}}$ takes $T_{(x,X)}\mathcal{M}_k$ to $T_{(x,X)}\mathcal{W}_k$. It also follows from the proof of Theorem 3.5 that the orthogonal projection from $\overline{T_{(x,X)}\mathcal{W}_k}$ to $\overline{T_{(x,X)}G\cdot(x,X)}$ with respect to $\bar{\mathrm{g}}$ takes $T_{(x,X)}\mathcal{W}_k$ onto $T_{(x,X)}G\cdot(x,X)$. Now let us remark that the complex structures I_j, $j = 1, 2, 3$, extend to complex structures of $L^2(H_+, H) \times L^2(H_+, H)$ by the same formulas, making $L^2(H_+, H) \times L^2(H_+, H)$ into a hyperkähler space. Since, for $j = 1, 2, 3$, the complex structure I_j fixes $T_{(x,X)}\mathcal{M}_k$ and is orthogonal with respect to $\bar{\mathrm{g}}$, it follows that for $j = 1, 2, 3$, the orthogonal projection from $L^2(H_+, H) \times L^2(H_+, H)$ onto $I_j\left(\overline{T_{(x,X)}G\cdot(x,X)}\right)$ takes $T_{(x,X)}\mathcal{M}_k$ onto $I_j\left(T_{(x,X)}G\cdot(x,X)\right)$. Hence, from the orthogonal sum

$$\begin{aligned} L^2(H_+, H) \times L^2(H_+, H) &= \overline{T_{(x,X)}G\cdot(x,X)} \oplus \overline{H_{(x,\,X)}} \oplus I_1\left(\overline{T_{(x,X)}G\cdot(x,X)}\right) \\ &\quad \oplus I_2\left(\overline{T_{(x,X)}G\cdot(x,X)}\right) \oplus I_3\left(\overline{T_{(x,X)}G\cdot(x,X)}\right), \end{aligned}$$

it follows that

$$\begin{aligned} T_{(x,X)}\mathcal{M}_k &= T_{(x,X)}G\cdot(x,X) \oplus H_{(x,X)} \oplus I_1\left(T_{(x,X)}G\cdot(x,X)\right) \\ &\qquad \oplus I_2\left(T_{(x,X)}G\cdot(x,X)\right) \oplus I_3\left(T_{(x,X)}G\cdot(x,X)\right). \end{aligned}$$

□

4 A 1-parameter family of hyperkähler structures on the cotangent bundle of the restricted Grassmannian

4.1 The stable manifold $\mathcal{W}_k^{s_1}$ associated with the complex structure I_1

Recall that the complex Banach Lie group $G^{\mathbb{C}} := GL(H_+) \cap \{\ \mathrm{Id} + L^1(H_+)\ \}$ acts I_1-holomorphically on $T\mathcal{M}_k$ by

$$g\cdot((x,X)) = (x \circ g^{-1}, X \circ g^*),$$

for all g in $G^{\mathbb{C}}$, and for all (x,X) in $T\mathcal{M}_k$. Let $\mathcal{W}_k^{s_1}$ be the stable manifold associated with $\mathcal{W}_k$ with respect to the complex structure I_1, i.e. the union of $G^{\mathbb{C}}$-orbits (for the above action) intersecting $\mathcal{W}_k$. Since by Proposition 3.7, assumption (S) is satisfied, one has the following Corollary of Lemma 2.24 and Theorem 2.25:

Corollary 4.1 *The space $\mathcal{W}_k^{s_1}$ is a I_1-complex submanifold of $T\mathcal{M}_k$ and the quotient space $\mathcal{W}_k^{s_1}/G^{\mathbb{C}}$ is a smooth complex manifold. The map from $\mathcal{W}_k/G$ to $\mathcal{W}_k^{s_1}/G^{\mathbb{C}}$ induced by the natural injection of $\mathcal{W}_k$ into $\mathcal{W}_k^{s_1}$ is an I_1-holomorphic diffeomorphism.* □

In the following Proposition we give an explicit characterization of the stable manifold $\mathcal{W}_k^{s_1}$ and we compute the projection :

$$\begin{array}{rccl} q_1: & \mathcal{W}_k^{s_1} & \to & \mathcal{W}_k \\ & (x,X) & \mapsto & q_1((x,X)) = g_{(x,X)}\cdot(x,X). \end{array}$$

defined by Proposition 2.19.

Proposition 4.2 *The stable manifold $\mathcal{W}_k^{s_1}$ is the set*

$$\{(x,X) \in T\mathcal{M}_k \textit{ such that } X^*x = 0 \textit{ and } x \textit{ is one-to-one}\},$$

and, for all (x,X) *in* $\mathcal{W}_k^{s_1}$*, the unique element* $g_{(x,X)}$ *of* $\exp i\mathfrak{g}$ *such that* $g_{(x,X)} \cdot (x,X)$ *belongs to* $\mathcal{W}_k$ *is defined by*

$$g_{(x,X)}^{-1} := \left(\frac{k^2}{2}(x^*x)^{-1} + \frac{k^2}{2}(x^*x)^{-\frac{1}{2}} \left(Id_{H_+} + \frac{4}{k^4}(x^*x)^{\frac{1}{2}} X^* X (x^*x)^{\frac{1}{2}} \right)^{\frac{1}{2}} (x^*x)^{-\frac{1}{2}} \right)^{\frac{1}{2}}.$$

□ **Proof of Proposition** 4.2 :

Let $\mathcal{A}$ be the set of elements (x,X) in $T\mathcal{M}_k$ such that $X^*x = 0$ and x is one-to-one. Let us show that $\mathcal{W}_k^{s_1} \subset \mathcal{A}$. Consider $(x,X) \in \mathcal{W}_k^{s_1}$ and $g \in G^{\mathbb{C}}$ such that $g \cdot (x,X) \in \mathcal{W}_k$. We have:

$$(X \circ g^*)^*(x \circ g^{-1}) = gX^*xg^{-1} = 0.$$

If $g^{-1} = |g|^{-1}.u^{-1}$ denotes the polar decomposition of g^{-1}, the equality:

$$(x \circ g^{-1})^*(x \circ g^{-1}) - (X \circ g^*)^*(X \circ g^*) = k^2 Id_{H_+},$$

reads:

$$|g|^{-1}x^*x|g|^{-1} - |g|X^*X|g| = k^2 Id_{H_+}.$$

Thus $|g|^{-1}x^*x|g|^{-1} = k^2 Id_{H_+} + |g|X^*X|g|$ is a positive definite self-adjoint operator. Since $|g|^{-1}$ is an isomorphism, the same holds true for x^*x. It follows that x is one-to-one.

To see that $\mathcal{A} \subset \mathcal{W}_k^{s_1}$, consider an element (x,X) of $\mathcal{A}$. We are looking for a positive definite self-adjoint operator $g_{(x,X)}$ such that :

$$g_{(x,X)}^{-1} x^* x g_{(x,X)}^{-1} - g_{(x,X)} X^* X g_{(x,X)} = k^2 Id_{H_+}.$$

The operator x being one-to-one, x^*x is positive definite and its square root $(x^*x)^{\frac{1}{2}}$ is an invertible operator on H_+. Hence it is sufficient to find a positive definite operator $\gamma := (x^*x)^{\frac{1}{2}} g_{(x,X)}^{-1}$ such that:

$$\gamma^*\gamma - \gamma^{-1}(x^*x)^{\frac{1}{2}} X^* X (x^*x)^{\frac{1}{2}} \gamma^{-1*} = k^2 Id_{H_+}$$

$$\Leftrightarrow (\gamma\gamma^*)^2 - k^2(\gamma\gamma^*) - (x^*x)^{\frac{1}{2}} X^* X (x^*x)^{\frac{1}{2}} = 0$$

The unique positive definite solution of the latter equation is :

$$\gamma\gamma^* = \frac{k^2}{2}\left(Id_{H_+} + \left(Id_{H_+} + \frac{4}{k^4}(x^*x)^{\frac{1}{2}} X^* X (x^*x)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right).$$

Therefore

$$g_{(x,X)}^{-2} := (x^*x)^{-\frac{1}{2}} \gamma\gamma^* (x^*x)^{-\frac{1}{2}} \tag{7}$$

is positive definite and self-adjoint, and its square root satisfies the required condition. □

4.2 Identification of $\mathcal{W}_k^{s_1}/G^{\mathbb{C}}$ with the cotangent space $T'Gr_{res}^0$ of the restricted Grassmannian

In this Subsection , we will use the following Theorem to identify the complex quotient space $\mathcal{W}_k^{s_1}/G^{\mathbb{C}}$ with the cotangent bundle $T'Gr_{res}^0$ of the connected component Gr_{res}^0 of the restricted Grassmannian. Recall that Ran x denotes the range of an operator x.

Theorem 4.3 *The map Ψ defined by*

$$\begin{array}{cccc} \Psi \ : & \mathcal{W}_k^{s_1} & \longrightarrow & T'Gr_{res}^0 \\ & (x, X) & \longmapsto & (\operatorname{Ran} x,\ \frac{1}{k^2} x \circ X^*) \end{array}$$

is an I_1-holomorphic submersion whose fibers are the orbits under the I_1-holomorphic action of the complex group $G^{\mathbb{C}}$ on $\mathcal{W}_k^{s_1} \subset T\mathcal{M}_k$.

■ **Proof of Theorem** 4.3:

. For $(x,\, X)$ in $\mathcal{W}_k^{s_1}$, the range P of x is an element of Gr_{res}^0 since $p_+ \circ x$ belongs to $\{Id_{H_+} + L^1(H_+)\}$, thus is a Fredholm operator with index 0, and $p_- \circ x$ is a Hilbert-Schmidt operator. Furthermore the condition $X^* x = 0$ implies that the restriction of $\eta \ := \frac{1}{k^2} x \circ X^*$ to P vanishes. Thus η can be identified with an element of $L^2(P^\perp, P)$ which is the cotangent space of Gr_{res}^0 at P.

. Let us check that Ψ is onto. For P in Gr_{res}^0, denote by x_P the operator from H_+ to H whose columns are the vectors of the canonical basis of P as defined in [38]. Then $k\, x_P$ is in $\mathcal{M}_k$ (see [47] for the details of this affirmation). On the other hand, for every $V \in L^2(P^\perp, P)$, the operator X defined by $X \ := k^2\ V^* \circ x_P^{*-1}$ (where x_P is viewed as an isomorphism between H_+ and P) satisfies $\frac{1}{k^2} x_P \circ X^* = V$ and is an element of $L^2(H_+, P^\perp)$. Moreover, since $p_- \ : P \to H_-$ is Hilbert-Schmidt, $p_+ \ : P^\perp \to H_+$ is also Hilbert-Schmidt and it follows that $p_+ \circ X \in L^1(H_+)$. Thus $\Psi\left((kx_P,\ k^2\ V^* \circ x_P^{*-1})\right) = (P, V)$.

. Let us show that two elements (x_1, X_1) and (x_2, X_2) in $\mathcal{W}_k^{s_1}$ have the same image by Ψ if and only if they are in the same orbit under $G^{\mathbb{C}}$. We have:

$$\text{Ran } x_1 = \text{ Ran } x_2 \Leftrightarrow x_2 = x_1 \circ g^{-1} \text{ for some } g^{-1} \in G^{\mathbb{C}},$$

thus :

$$x_2 \circ X_2^* = x_1 \circ X_1^* = x_2 \circ g \circ X_1^*,$$

which is equivalent to : $X_2 = X_1 \circ g^*$ since x_2 is one-to-one.

. Let us explicit the differential of Ψ at $(x,\, X)$. Denote by P the range of x, by $\mathcal{U}_P \subset Gr_{res}^0$ the open subset of elements $P' \in Gr_{res}^0$ such that the orthogonal projection of P' onto P is an isomorphism and by φ_P the chart from $\mathcal{U}_P$ onto $L^2(P, P^\perp)$ which maps P' to the unique element U in $L^2(P, P^\perp)$ whose graph is P'. Let

$$(Z, T) \in T_{(x,X)}\mathcal{W}_k^{s_1}$$

and

$$(x(t), X(t)) \in \mathcal{C}^1\left((-\epsilon, \epsilon), \mathcal{W}_k^{s_1}\right)$$

be such that :

$$\dot{x}(0) = Z \text{ and } \dot{X}(0) = T.$$

Denote by $(U(t), V(t))$ the curve $\varphi \circ \Psi((x(t), X(t))$. Since

$$\text{Ran } x(t) = \text{ Ran } (\text{ Id}_P + U(t))$$

and since $U(0) = 0$, there exists $g(t) \in \mathcal{C}^1((-\epsilon, \epsilon), G^{\mathbb{C}})$ such that

$$x(t) \circ g(t)^{-1} = \text{ Id}_P + U(t) \text{ and } x(0) \circ g(0)^{-1} = \text{ Id}_P.$$

Considering the decomposition of H into $P \oplus P^\perp$, one has : $g(t) = pr_P \circ x(t)$, where pr_P denotes the orthogonal projection onto P. Thus $U(t) = pr_{P^\perp} \circ x(t) \circ (pr_P \circ x(t))^{-1}$ and

$$\frac{d}{dt}_{|t=0} U(t) = pr_{P^\perp} \circ Z \circ x(0)^{-1}.$$

Moreover one has $V(t) = \frac{1}{k^2} pr_P \circ x(t) \circ X(t)^*_{|P^\perp}$ and

$$\frac{d}{dt}_{|t=0} V(t) = \frac{1}{k^2}\left(pr_P(Z) \circ X^* + x \circ pr_{P^\perp}(T)^*\right).$$

Therefore

$$d\varphi_P \circ d\Psi_{(x,X)}((Z, T)) = \left(pr_{P^\perp} \circ \ Z \ \circ x^{-1}, \ \frac{1}{k^2}\left(pr_P(Z) \ \circ \ X^* + x \ \circ \ pr_{P^\perp}(T)^*\right)\right).$$

It follows that :

$$d\varphi_P \circ d\Psi_{(x,X)}(I_1(Z, T)) = i d\varphi_P \circ d\Psi_{(x,X)}((Z, T))$$

thus Ψ is I_1-holomorphic. Furthermore $d\varphi_P \circ d\Psi_{(x,X)}$ is surjective, a preimage of $(U, V) \in L^2(P, P^\perp) \times L^2(P^\perp, P)$ being given by $(U \circ x, k^2 V^* x^{*-1})$. At last, from the above considerations it follows that the kernel of $d\Psi_{(x,\, X)}$ is the tangent space of the $G^{\mathbb{C}}$-orbit $G^{\mathbb{C}} \cdot (x,\, X)$, which splits by Proposition 3.7. ■

Corollary 4.4 *The quotient space $\mathcal{W}_k^{s_1}/G^{\mathbb{C}}$ is isomorphic as a smooth complex manifold to the cotangent space $T'Gr_{res}^0$ endowed with its natural complex structure via the following isomorphism*

$$\begin{array}{rcccl} \tilde{\Psi} & : & \mathcal{W}_k^{s_1}/G^{\mathbb{C}} & \longrightarrow & T'Gr_{res}^0 \\ & & [(x,X)] & \longmapsto & (\mathrm{Ran}\, x, \frac{1}{k^2} x \circ X^*). \end{array}$$

Hence $T'Gr_{res}^0$ carries a 1-parameter family of hyperkähler structures indexed by $k \in \mathbb{R}^*$.

Remark 4.5 By exchanging H_+ with a subspace of another connected component of Gr_{res}, we obtain the cotangent space of every connected component of Gr_{res} as a hyperkähler quotient.

By restriction to the zero section of the tangent space $T\mathcal{M}_k$ one deduces from the previous Theorem the following result, which has been, as already mentioned in the Introduction, partially obtained by T. Wurzbacher (cf [51]) :

Corollary 4.6 *For every $k \in \mathbb{R}^*$, the connected component Gr_{res}^0 of the restricted Grassmannian is diffeomorphic to the Kähler quotient of the space $\mathcal{M}_k$ by the Hamiltonian action of the unitary group G, with level set*

$$\{\ x \in \mathcal{M}_k\, ,\ x^*x \ = \ k^2\ Id\ \}.$$

4.3 The Kähler potential K_1 of $T'Gr_{res}^0$

The hyperkähler manifold $T\mathcal{M}_k$ admits a globally defined hyperkähler potential, i.e. a Kähler potential with respect to all complex structures, which has the following expression :

$$\begin{array}{rcccl} K & : & T\mathcal{M}_k & \to & \mathbb{R} \\ & & (x,X) & \mapsto & \frac{1}{4}\ \mathrm{Tr}\ (x^*x + X^*X - k^2 Id). \end{array}$$

The theory of Subsection 2.4 applied to the particular case of $T'Gr_{res}^0$ yields the following Theorem :

Theorem 4.7 *For all $\frac{k^2}{2} \in \mathbb{N}^*$, the 2-form $\Psi^*\omega_1^{red}$ on $\mathcal{W}_k^{s_1}$ satisfies $i\Psi^*\omega_1^{red} = dd^{c_1}K_1$, where for all (x,X) in $\mathcal{W}_k^{s_1}$,*

$$K_1\left((x,\,X)\right) = \frac{k^2}{4} \log\det\left(\frac{x^*x}{k^2}\right) + \frac{k^2}{2}\ Tr\ \left(\frac{\gamma\gamma^*}{k^2} -\ Id\ \right) - \frac{k^2}{4}\ Tr\ \left(\log\frac{\gamma\gamma^*}{k^2}\right),$$

with $\gamma\gamma^* := \frac{k^2}{2}\left(Id_{H_+} + \left(Id_{H_+} + \frac{4}{k^4}(x^*x)^{\frac{1}{2}} X^*X (x^*x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right).$

■ **Proof of Theorem** 4.7:
By Theorem 2.34, one has :

$$K_1((x,X)) \ := K\left(g_{(x,X)}\cdot(x,X)\right) + \frac{1}{2}\log|\,\chi_{\frac{k^2}{2}}(g_{(x,X)})\,|^2.$$

Since $g_{(x,X)}X^*Xg_{(x,X)} = g^{-1}_{(x,X)}x^*xg^{-1}_{(x,X)} - k^2\ \mathrm{Id}$, one has :

$$\begin{aligned} K\left(g_{(x,X)}\cdot(x,X)\right) \ &:= \tfrac{1}{4}\ \mathrm{Tr}\ \left(g^{-1}_{(x,X)}x^*xg^{-1}_{(x,X)} + g_{(x,X)}X^*Xg_{(x,X)} - k^2\ \mathrm{Id}\right) \\ &= \tfrac{1}{2}\ \mathrm{Tr}\ \left(g^{-1}_{(x,X)}x^*xg^{-1}_{(x,X)} - k^2\ \mathrm{Id}\right). \end{aligned}$$

Thus, after conjugation by $g^{-1}_{(x,\,X)}$:

$$K\left(g_{(x,X)}\cdot(x,X)\right) = \frac{k^2}{2}\ \mathrm{Tr}\ \left(g^{-2}_{(x,X)}\frac{x^*x}{k^2} -\ \mathrm{Id}\right),$$

and by (7)

$$g^{-2}_{(x,X)}x^*x = (x^*x)^{-\frac{1}{2}}\gamma\gamma^*(x^*x)^{\frac{1}{2}}.$$

After conjugation by $(x^*x)^{-\frac{1}{2}}$, we have :

$$K\left(g_{(x,X)}.(x,X)\right) = \frac{k^2}{2}\ \mathrm{Tr}\ \left(\frac{\gamma\gamma^*}{k^2} -\ \mathrm{Id}\right).$$

On the other hand,

$$\begin{aligned} \tfrac{1}{2}\log|\chi_{\frac{k^2}{2}}\left(g_{(x,X)}\right)|^2 \ &= -\tfrac{1}{2}\log\left(\det\left(g^{-2}_{(x,X)}\right)\right)^{\frac{k^2}{2}} \\ &= -\tfrac{k^2}{4}\log\det{(x^*x)^{-\frac{1}{2}}}\ \gamma\gamma^*(x^*x)^{-\frac{1}{2}} \\ &= \tfrac{k^2}{4}\log\det\left(\tfrac{x^*x}{k^2}\right) - \tfrac{k^2}{4}\log\det\left(\tfrac{\gamma\gamma^*}{k^2}\right). \end{aligned}$$

Furthermore, the operator

$$A \ := \frac{\gamma\gamma^*}{k^2} -\ \mathrm{Id}\ = \frac{1}{2}\left(\left(\ \mathrm{Id}\ + \frac{4}{k^4}(x^*x)^{\frac{1}{2}}X^*X(x^*x)^{\frac{1}{2}}\right)^{\frac{1}{2}} -\ \mathrm{Id}\ \right)$$

is self-adjoint positive and of trace class. Thus :

$$\log\det\left(\frac{\gamma\gamma^*}{k^2}\right) = \ \mathrm{Tr}\ \log\left(\frac{\gamma\gamma^*}{k^2}\right).$$

■

Proposition 4.8 *For every $\frac{k^2}{2} \in \mathbb{N}^*$, the 2-form $\Psi^*\omega_1^{red}$ on $\mathcal{W}_k^{s_1}$ satisfies $i\Psi^*\omega_1^{red} = dd^{c_1}K_1$, with :*

$$\begin{aligned} K_1\left((x,X)\right) &= \tfrac{k^2}{4}\log\det\left(\tfrac{x^*x}{k^2}\right) + \tfrac{k^2}{4}\ Tr\ \Big(\left(\ Id\ + 4V^*V\right)^{\frac{1}{2}} - \ Id\ \Big) \\ &\quad -\tfrac{k^2}{4}\ Tr\ \log\tfrac{1}{2}\Big(\ Id\ + \left(\ Id\ + 4V^*V\right)^{\frac{1}{2}}\Big) \end{aligned}$$

where $V^* = \frac{1}{k^2}x \circ X^*$ is the image of the class $[(x,X)]$ under the identification $\tilde{\Psi}\ : \mathcal{W}_k^{s_1}/G^{\mathbb{C}} \to T'Gr_{res}$ given by Corollary 4.4 .

□ **Proof of Proposition** 4.8:
Since :

$$\frac{\gamma\gamma^*}{k^2} = \frac{1}{2}\left(\mathrm{Id} + \left(\mathrm{Id} + \frac{4}{k^4}|x|X^*X|x|\right)^{\frac{1}{2}}\right),$$

the operator $\frac{\gamma\gamma^*}{k^2}$ is conjugate to :

$$\frac{1}{2}\left(\mathrm{Id} + \left(\mathrm{Id} + \frac{4}{k^4}xX^*Xx^*\right)^{\frac{1}{2}}\right).$$

Hence one has :

$$\begin{aligned} \mathrm{Tr}\ \Big(\tfrac{\gamma\gamma^*}{k^2} - \ \mathrm{Id}\ \Big) - \tfrac{1}{2}\ \mathrm{Tr}\ \Big(\log\tfrac{\gamma\gamma^*}{k^2}\Big) &= \mathrm{Tr}\ \Big(\left(\ \mathrm{Id}\ + 4V^*V\right)^{\frac{1}{2}} - \ \mathrm{Id}\ \Big) \\ &\quad -\tfrac{1}{2}\ \mathrm{Tr}\ \log\tfrac{1}{2}\Big(\ \mathrm{Id}\ + \left(\ \mathrm{Id}\ + 4V^*V\right)^{\frac{1}{2}}\Big). \end{aligned}$$

□

4.4 The Kähler potential K_1 as a function of the curvature of Gr_{res}^0

Theorem 4.9 *The potential K_1 has the following expression in terms of the curvature :*

$$K_1((x,X)) = \frac{k^2}{4}\log\det\left(\frac{x^*x}{k^2}\right) + k^2 \mathrm{g}_{Gr}\left(f(I_1 R_{I_1 V,V})V, V\right),$$

with $V = \frac{1}{k^2}X \circ x^*$ and $f(u) = \frac{1}{u}\left(\sqrt{1+u} - 1 - \log\frac{1+\sqrt{1+u}}{2}\right)$.

■ **Proof of Theorem** 4.9:
The Grassmannian Gr_{res}^0 is a Hermitian-symmetric orbit of the connected component $\mathcal{U}_{res}^0$ of the restricted unitary group. Its curvature is therefore given by (see [46]) :

$$R_{X,Y}Z = YX^*Z - ZY^*X + ZX^*Y - XY^*Z,$$

for all $X, Y, Z \in T_P Gr^0_{res}$. The operator $R_{I_1 V, V}$ acts on $T_P Gr^0_{res}$ by :

$$R_{I_1 V, V} Y = -2i(VV^*Y + YV^*V).$$

It follows that :

$$\begin{aligned} g_{Gr}(I_1 R_{I_1 V,V} V, V) &= 2\Re \text{ Tr } (V^*VV^*V + V^*VV^*V) = 4\Re \text{ Tr } ((V^*V)^2) \\ &= \tfrac{1}{4}\Re \text{ Tr } ((4V^*V)^2), \end{aligned}$$

and :

$$\begin{aligned} g_{Gr}\left((I_1 R_{I_1 V,V})^j V, V\right) &= \Re \text{ Tr } \left(4^j (V^*V)^{j+1}\right) \\ &= \tfrac{1}{4}\Re \text{ Tr } \left((4V^*V)^{j+1}\right). \end{aligned}$$

Therefore one has :

$$\begin{aligned} \tfrac{1}{4} \text{ Tr } \left((\text{ Id } + 4V^*V)^{\frac{1}{2}} - \text{ Id } \right) - \tfrac{1}{2} \text{ Tr } \log \tfrac{1}{2}\left(\text{ Id } + (\text{ Id } + 4V^*V)^{\frac{1}{2}}\right) \\ = g_{Gr}\left(f(I_1 R_{I_1 V,V})V, V\right), \end{aligned}$$

with

$$f(u) = \frac{1}{u}\left(\sqrt{1+u} - 1 - \log\frac{1+\sqrt{1+u}}{2}\right).$$

■

Remark 4.10 The first summand in the expression of K_1 is the pull-back to $\mathcal{W}_k^{s_1}$ of the Kähler potential of the restricted Grassmannian (defined on the stable manifold $\mathcal{M}_k^s$ of Gr^0_{res}) via the canonical injection $\mathcal{M}_k^s \hookrightarrow \mathcal{W}_k^{s_1}$. Note that the Kähler potential of Gr^0_{res} is the pull-back of the Kähler potential of the complex projective space of a separable Hilbert space by Plücker's embedding. The second summand is expressed as a function of the curvature of the restricted Grassmannian applied to the image of an element (x, X) in $\mathcal{W}_k^{s_1}$ by the identification $\mathcal{W}_k^{s_1}/G^{\mathbb{C}} = T'Gr^0_{res}$ given by Corollary 4.4.

5 A 1-parameter family of hyperkähler structures on a natural complexification of the restricted Grassmannian

5.1 Definition of the complexified orbit $\mathcal{O}^{\mathbb{C}}$ of Gr^0_{res}

Let $\mathcal{U}_2(H)$ be the Banach Lie group $\mathcal{U}(H) \cap \{\text{Id}_H + L^2(H)\}$. An element P of Gr^0_{res} can be identified with $ik^2 pr_P$, $k \neq 0$, where pr_P denotes the orthogonal projection of H onto P. Via this identification, the natural action of $\mathcal{U}_2(H)$ on Gr^0_{res} is given by the conjugation. The complexified orbit $\mathcal{O}^{\mathbb{C}}$ of Gr_{res}^0 is the orbit of an element $P \in Gr_{res}^0$ under the action of the complex Lie group $GL_2(H) := GL(H) \cap \{\mathrm{Id}_H + L^2(H)\}$. It is the set of operators $z \in B(H)$ whose spectrum is the pair $\{ik^2\,,\,0\}$ with $k \neq 0$, and such that the eigenspace associated with ik^2 (resp. 0) is an element of Gr_{res}^0 (resp. of the Grassmannian Gr_{res}^{0*} obtained from the definition of Gr_{res}^0 by exchanging the roles of H_+ and H_-). This complexified orbit has been introduced in particular by J. Mickelsson in [32].

Proposition 5.1 *The complexified orbit $\mathcal{O}^{\mathbb{C}}$ of the connected component Gr_{res}^0 defined as the homogeneous space :*

$$\mathcal{O}^{\mathbb{C}} := GL_2(H)/\left(GL_2(H_+) \times GL_2(H_-)\right)$$

is a Hilbert manifold modelled over the Hilbert space $L^2(H_+, H_-) \times L^2(H_-, H_+)$, diffeomorphic to the open set of $Gr_{res}^0 \times Gr_{res}^{0*}$ consisting of all ordered pairs $(P, Q) \in Gr_{res}^0 \times Gr_{res}^{0*}$ such that $P \cap Q = \{0\}$.

□ **Proof of Proposition** 5.1:
Let us denote by ε the operator $ik^2 p_+$. The stabilizer of ε under the action of $GL_2(H)$ by conjugation is $GL_2(H_+) \times GL_2(H_-)$. The tangent space at ε of the homogeneous space $GL_2(H) \cdot \varepsilon$ is isomorphic to $\mathfrak{gl}_2(H)/\left(\mathfrak{gl}_2(H_+) \times \mathfrak{gl}_2(H_-)\right)$ which can be identified with $L^2(H_+, H_-) \times L^2(H_-, H_+)$. For $g \in GL_2(H)$, $g\varepsilon g^{-1} = ik^2 pr_{g.H_+}$, where $pr_{g.H_+}$ denotes the projection on $g.H_+$ parallel to $g.H_-$. Since g belongs to $GL_2(H)$, the orthogonal projection of $g.H_+$ to H_- is an Hilbert-Schmidt operator, and the orthogonal projection of $g.H_+$ to H_+ is a Fredholm operator of index 0. Similarly, the orthogonal projection of $g.H_-$ to H_+ is Hilbert-Schmidt, and the orthogonal projection of $g.H_-$ to H_- is a Fredholm operator of index 0. Thus $g.H_+$ belongs to Gr_{res}^0 and $g.H_-$ to Gr_{res}^{*0}. Moreover $g.H_+ \cap g.H_- = \{0\}$. □

5.2 The stable manifold $\mathcal{W}_k^{s_3}$ associated with the complex structure I_3

Recall that $I_3(Z, T) = (iT,\, iZ)$ for $(Z, T) \in T_{(x,X)}T\mathcal{M}_k$, and that the action of the Lie algebra $\mathfrak{g}$ on $(x, X) \in T\mathcal{M}_k$ is given by

$$\mathfrak{a}.(x, X) = (-x \circ \mathfrak{a}, -X \circ \mathfrak{a}),$$

for all $\mathfrak{a}$ in $\mathfrak{g}$. The action $\cdot_3$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ on $T\mathcal{M}_k$ compatible with I_3 is defined by

$$i\mathfrak{a}\cdot_3(x, X) := I_3\left(\mathfrak{a}.(x, X)\right) = (-iX \circ \mathfrak{a}, -ix \circ \mathfrak{a}) = (x, X)\begin{pmatrix} 0 & -i\mathfrak{a} \\ -i\mathfrak{a} & 0 \end{pmatrix},$$

for all $\mathfrak{a}$ in $\mathfrak{g}$. This action integrates into an I_3-holomorphic action of $G^{\mathbb{C}} = \exp i\mathfrak{g}\; G$ on $T\mathcal{M}_k$, also denoted by $\cdot_3$, and given by

$$\exp(i\mathfrak{a})u \cdot_3 (x, X) := (x \circ u^{-1}\,,\, X \circ u^{-1}) \begin{pmatrix} \cosh i\mathfrak{a} & -\sinh i\mathfrak{a} \\ -\sinh i\mathfrak{a} & \cosh i\mathfrak{a} \end{pmatrix},$$

for all $\mathfrak{a}$ in $\mathfrak{g}$ and for all u in G. By Lemma 2.24, the stable manifold associated with I_3 is the I_3-complex submanifold of $T\mathcal{M}_k$, contained in $\mu_1^{-1}\left(-\frac{i}{2}k^2\ \mathrm{Tr}\ \right) \cap \mu_2^{-1}(0)$, defined as

$$\mathcal{W}_k^{s_3} := \{\ (x, X) \in T\mathcal{M}_k,\ \ \exists\ \mathfrak{a} \in \mathfrak{g},\ \exp i\mathfrak{a} \cdot_3 (x, X) \in \mathcal{W}_k\ \}\,.$$

5.3 Identification of $\mathcal{W}_k^{s_3}/G^{\mathbb{C}}$ with $\mathcal{O}^{\mathbb{C}}$

Theorem 5.2 *The map ψ defined by :*

$$\begin{array}{rccl} \psi : & \mathcal{W}_k^{s_3} & \to & \mathcal{O}^{\mathbb{C}} \\ & (x, X) & \mapsto & z = i(x + X)(x^* - X^*) \end{array}$$

is an I_3-holomorphic submersion whose fibers are the orbits of the I_3-holomorphic action of $G^{\mathbb{C}}$ on $T\mathcal{M}$.

■ **Proof of Theorem** 5.2:

. Let us show that ψ takes all its values in $\mathcal{O}^{\mathbb{C}}$. Recall that, for $(x, X) \in \mathcal{W}_k^{s_3}$, one has :

$$x^*x - X^*X = k^2\ \mathrm{Id} \quad \text{and} \quad X^*x = x^*X.$$

Thus :

$$\begin{array}{l} (x^* - X^*)(x + X) = k^2\ \mathrm{Id}, \\ (x^* + X^*)(x - X) = k^2\ \mathrm{Id}. \end{array} \tag{8}$$

It follows that $\mathrm{Ker}\ (x + X) = \{0\}$ and $\mathrm{Ker}\ (x - X) = \{0\}$. The kernel of z is therefore :

$$\mathrm{Ker}\ z \ = \ \mathrm{Ker}\ (x^* - X^*) \ = \ (\mathrm{Ran}\,(x - X))^{\perp}\,.$$

Moreover, since for all $v \in H$:

$$z\left((x + X)v\right) = i(x + X)(x^* - X^*)(x + X)v = ik^2(x + X)v,$$

the subspace $\mathrm{Ran}\,(x + X)$ is contained in the eigenspace of z corresponding to the eigenvalue ik^2. Hence :

$$\mathrm{Ran}\,(x + X) \quad \cap \quad \mathrm{Ker}\ (x^* - X^*) = \{0\}.$$

Further, the projection of H onto $\mathrm{Ran}\,(x + X)$ is given by

$$\begin{array}{rccl} p_1 : & H & \to & \mathrm{Ran}(x + X) \\ & v & \mapsto & \frac{1}{k^2}(x + X)(x^* - X^*)v \end{array}$$

and is continuous. Since $\mathrm{Id}_H - p_1$ takes its values in $\mathrm{Ker}\,(x^* - X^*)$, one has

$$\mathrm{Ran}(x+X) \oplus \mathrm{Ker}\,(x^* - X^*) = H$$

as a direct topological sum. Moreover, for $(x, X) \in T\mathcal{M}_k$, $\mathrm{Ran}(x + X)$ and $\mathrm{Ran}(x - X)$ are elements of Gr^0_{res}, thus $\mathrm{Ker}\,(x^* - X^*) = (\mathrm{Ran}(x - X))^\perp$ is an element of Gr^{0*}_{res}. It follows that ψ takes values in $\mathcal{O}^{\mathbb{C}}$.

. Let us show that the fibers of ψ are the orbits under the I_3-holomorphic action of $G^{\mathbb{C}}$. Suppose that $\psi\left((x_1, X_1)\right) = \psi\left((x_2, X_2)\right)$ where (x_1, X_1) and (x_2, X_2) are in $\mathcal{W}_k^{s_3}$. It follows that :

$$\mathrm{Ran}(x_1 + X_1) = \mathrm{Ran}(x_2 + X_2) \quad \text{and} \quad \mathrm{Ran}(x_1 - X_1) = \mathrm{Ran}(x_2 - X_2).$$

Therefore there exists $g \in GL(H_+)$ such that $(x_2 + X_2) = (x_1 + X_1) \circ g$ and $g' \in GL(H_+)$ such that $(x_2 - X_2) = (x_1 - X_1) \circ g'$. This implies that:

$$\begin{aligned} 2x_2 &= x_1(g + g') + X_1(g - g') \\ 2X_2 &= x_1(g - g') + X_1(g + g') \end{aligned}$$

Recall that for $i = 1, 2$, $x_i^* x_i - X_i^* X_i = k^2\,\mathrm{Id}$ and $X_i^* x_i = x_i^* X_i$. We have:

$$\begin{aligned} 4(x_2^* x_2 - X_2^* X_2) &= (g^* + g'^*)(x_1^* x_1 - X_1^* X_1)(g + g') \\ &+ (g^* - g'^*)(X_1^* X_1 - x_1^* x_1)(g - g') \\ &+ (g^* + g'^*)(x_1^* X_1 - X_1^* x_1)(g - g') \\ &+ (g^* - g'^*)(X_1^* x_1 - x_1^* X_1)(g + g'), \end{aligned}$$

i.e.

$$g^* g' + g'^* g = 2\,\mathrm{Id},$$

and

$$\begin{aligned} 4(X_2^* x_2 - x_2^* X_2) &= (g^* - g'^*)(x_1^* x_1 - X_1^* X_1)(g + g') \\ &+ (g^* + g'^*)(X_1^* X_1 - x_1^* x_1)(g - g') \\ &+ (g^* - g'^*)(x_1^* X_1 - X_1^* x_1)(g - g') \\ &+ (g^* + g'^*)(X_1^* x_1 - x_1^* X_1)(g + g'), \end{aligned}$$

that is :

$$g^* g' = g'^* g.$$

Thus $g' = g^{*-1}$. Denoting by $\exp(-i\mathfrak{a}).u^{-1} = g^{-1}$ the polar decomposition of g^{-1}, with $u \in \mathcal{U}(H_+)$ and $\mathfrak{a} \in \mathfrak{u}(H_+)$, it follows that:

$$\begin{aligned} x_2 &= x_1 \cosh(i\mathfrak{a})u + X_1 \sinh(i\mathfrak{a})u \\ X_2 &= -x_1 \sinh(i\mathfrak{a})u + X_1 \cosh(i\mathfrak{a})u. \end{aligned}$$

Consequently : $(x_2, X_2) = \exp(-i\mathfrak{a})u^{-1} \cdot (x_1, X_1)$, in other words (x_1, X_1) and (x_2, X_2) belong to the same $G^{\mathbb{C}}$-orbit.

. Let us show that ψ is onto. Let $P \in Gr_{res}^0$ and $Q \in Gr_{res}^{0*}$ be such that $P \cap Q = \{0\}$. $Q^\perp$ is the graph of a Hilbert-Schmidt operator $A : P \to P^\perp$ and Q is the graph of $-A^* : P^\perp \to P$. Let f be the map that takes an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H_+ to the associated canonical basis of P and that takes an orthonormal basis $\{e_{-i}\}_{i\in\mathbb{N}^*}$ of H_- to the associated canonical basis of $P^\perp$. Denote by g the unitary element $f \circ |f|^{-1}$. Let us remark that g belongs to $\mathcal{U}_2(H)$ and satisfies : $p_+ \circ g_{|H_+} \in \mathrm{Id}_{H_+} + L^1(H_+)$ as well as $p_- \circ g_{|H_-} \in \mathrm{Id}_{H_-} + L^1(H_-)$. Define :

$$\begin{cases} x &= k(\mathrm{Id}_P + \frac{1}{2}A) \circ g_{|H_+} \\ X &= -\frac{k}{2}A \circ g_{|H_+}. \end{cases}$$

One has: $\mathrm{Ran}\ (x + X) = \mathrm{Ran}\ (g_{|H_+}) = P$ and $\mathrm{Ran}\ (x - X) = \mathrm{Ran}\ (\mathrm{Id}_P + A) \circ g_{|H_+} = Q^\perp$. Let us check that $(x\,,\,X)$ is an element of $T\mathcal{M}$. Denote by :

$$\mathrm{Id}_H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the block decomposition of the identity operator with respect to the direct sums $H = P \oplus P^\perp$ and $H = H_+ \oplus H_-$, where a (resp. d) belongs to $\mathrm{Fred}(P, H_+)$ (resp. $\mathrm{Fred}(P^\perp, H_-)$) and where b (resp. c) belongs to $L^2(P^\perp, H_+)$ (resp. $L^2(P, H_-)$). Further, denote by :

$$g = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

the block decomposition of g with respect to the directs sums $H = H_+ \oplus H_-$ and $H = P \oplus P^\perp$. The block decomposition of g with respect to $H = H_+ \oplus H_-$ is :

$$g = \begin{pmatrix} au_1 & bu_2 \\ cu_1 & du_2 \end{pmatrix}.$$

As mentioned above $p_+ \circ g_{|H_+} = au_1$ belongs to $\mathrm{Id}_{H_+} + L^1(H_+)$, and $p_- \circ g_{|H_-} = du_2$ belongs to $\mathrm{Id}_{H_-} + L^1(H_-)$. It follows that with respect to the direct sum $H = H_+ \oplus H_-$, the operator x has the following expression :

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k\ \mathrm{Id}_P \\ \frac{k}{2}A \end{pmatrix} \circ u_1 = \begin{pmatrix} kau_1 + \frac{k}{2}bAu_1 \\ kcu_1 + \frac{k}{2}dAu_1 \end{pmatrix}.$$

It follows that:

$$p_+ \circ x = kau_1 + \frac{k}{2}bAu_1 \in k\mathrm{Id}_{H_+} + L^1(H_+)$$

and

$$p_- \circ x = kcu_1 + \frac{k}{2} dAu_1 \in L^2(H_+, H_-).$$

Similarly,

$$p_+ \circ X = -\frac{k}{2} bAu_1 \in L^1(H_+)$$

and

$$p_- \circ X = -\frac{k}{2} dAu_1 \in L^2(H_+, H_-).$$

Hence the ordered pair (x, X) is in $T\mathcal{M}_k$. Besides, $x^*x - X^*X = k^2 \mathrm{Id}_{H_+}$ and $X^*x - x^*X = 0$. It remains to prove that $(x, X) \in \mathcal{W}_k^{s_3}$. For this purpose, observe that:

$$\begin{aligned} x^*x + X^*X &= k^2 \mathrm{Id}_{H_+} + \tfrac{k^2}{2} u_1^* A^* A u_1 \\ X^*x + x^*X &= -\tfrac{k^2}{2} u_1^* A^* A u_1. \end{aligned}$$

The condition $\exp i\mathfrak{a} \cdot_3 (x\,,\,X) \in \mathcal{W}_k$ is equivalent to the following equation :

$$\begin{aligned} \cosh i\mathfrak{a} \circ \left(\mathrm{Id}_{H_+} + \tfrac{1}{2} u_1^* A^* A u_1\right) \circ \sinh i\mathfrak{a} + \sinh i\mathfrak{a} \circ \left(\mathrm{Id}_{H_+} + \tfrac{1}{2} u_1^* A^* A u_1\right) \circ \cosh i\mathfrak{a} \\ + \cosh i\mathfrak{a} \circ \left(\tfrac{1}{2} u_1^* A^* A u_1\right) \circ \cosh i\mathfrak{a} + \sinh i\mathfrak{a} \circ \left(\tfrac{1}{2} u_1^* A^* A u_1\right) \circ \sinh i\mathfrak{a} &= 0, \end{aligned}$$

whose solution is:

$$\mathfrak{a} = \frac{i}{4} \log\left(\mathrm{Id}_{H_+} + u_1^* A^* A u_1\right),$$

which belongs to $\mathcal{A}^1(H_+)$.

. The differential of ψ at an element $(x, X) \in \mathcal{W}_k^{s_3}$ maps the ordered pair $(Z, T) \in T_{(x,X)}\mathcal{W}_k^{s_3}$ to:

$$d\psi_{(x,X)}\left((Z, T)\right) = i(Z + T)(x^* - X^*) + i(x + X)(Z^* - T^*).$$

One has $d\psi_{(x,X)}(I_3\left((Z\,,\,T)\right) = id\psi_{(x,X)}\left((Z\,,\,T)\right)$, thus ψ is holomorphic. Let z be in $\mathcal{O}^{\mathbb{C}}$, and let P (resp. Q) be the eigenspace of z with respect to the eigenvalue ik^2 (resp. 0). Let $(U\,,\,V)$ be an element of $L^2(P\,,\,P^\perp) \times L^2(Q\,,\,Q^\perp)$. A preimage of $(U\,,\,V)$ by $d\psi_{(x,X)}$ is given by the ordered pair $(Z\,,\,T) \in L^2(H_+, H) \times L^2(H_+, H)$ defined by

$$\begin{aligned} U &= i(Z + T)(x^* - X^*) \\ V &= i(x + X)(Z^* - T^*). \end{aligned}$$

Using equations (8) page 45, one gets :

$$\begin{aligned} (Z + T) &= -\tfrac{i}{k^2} U(x + X) \\ (Z - T) &= \tfrac{i}{k^2} V^*(x - X). \end{aligned}$$

Hence :

$$\begin{array}{lll} Z & = -\frac{i}{2k^2}\left(U(x+X) - V^*(x-X)\right) \\ T & = -\frac{i}{2k^2}\left(U(x+X) + V^*(x-X)\right). \end{array}$$

Moreover, for $(x\,,\,X) \in \mathcal{W}_k^{s_3}$, $p_+ \circ Z$ and $p_+ \circ T$ are trace class operators. It follows that the differential $d\psi_{(x,X)}$ is onto. ■

Corollary 5.3 *The quotient space $\mathcal{W}_k^{s_3}/G^{\mathbb{C}}$ is diffeomorphic to the complexified orbit $\mathcal{O}^{\mathbb{C}}$ via the isomorphism*

$$\begin{array}{llll} \tilde{\psi}: & \mathcal{W}_k^{s_3}/G^{\mathbb{C}} & \rightarrow & \mathcal{O}^{\mathbb{C}} \\ & [(x,X)] & \mapsto & z = i(x+X)(x^* - X^*) \end{array}$$

□

5.4 The Kähler potential $\hat{K}_3$ of $T'Gr^0_{res}$

From the general theory it follows that:

$$\psi^*\omega_3^{red}((x,X)) = dd^{c_3}K(q_3(x,X)),$$

where: $K((x,X)) = \frac{1}{4}$ Tr $(x^*x + X^*X - k^2\mathrm{Id})$, and where q_3 is the projection from $\mathcal{W}_k^{s_3}$ to $\mathcal{W}_k$. This Subsection is devoted to the computation of the Kähler potential $K_3 := K \circ q_3$ associated with the complex structure I_3 at a point $(x\,,\,X)$ of the stable manifold $\mathcal{W}_k^{s_3}$ by the use of an invariant of the $G^{\mathbb{C}}$-orbits. We will use the following notation. The injection of $\mathfrak{g}$ into $\mathfrak{g}'$ given by the trace allows to identify the moment map μ_3 with the map (still denoted by μ_3) defined by

$$\mu_3\left((x,X)\right) = \frac{i}{2}(X^*x + x^*X).$$

Define a function μ_4 by :

$$\mu_4\left((x,X)\right) := \frac{i}{2}(x^*x + X^*X).$$

We have the following Lemma :

Lemma 5.4 *For every $(x\,,\,X)$ in $\mathcal{W}_k^{s_3}$, one has*

$$K_3\left((x,X)\right) = \frac{1}{4}\ Tr\ \left(\left(\mu_4^2\left((x,X)\right) - \mu_3^2\left((x,X)\right)\right)^{\frac{1}{2}} - k^2 Id\right).$$

△ **Proof of Lemma** 5.4:
One has :

$$\begin{array}{lll} \mu_3\left(\exp i\mathfrak{a} \cdot_3 (x,X)\right) & = & \cosh i\mathfrak{a} \circ \mu_4\left((x,X)\right) \circ \sinh i\mathfrak{a} + \sinh i\mathfrak{a} \circ \mu_4\left((x,X)\right) \circ \cosh i\mathfrak{a} \\ & & + \cosh i\mathfrak{a} \circ \mu_3\left((x,X)\right) \circ \cosh i\mathfrak{a} + \sinh i\mathfrak{a} \circ \mu_3\left((x,X)\right) \circ \sinh i\mathfrak{a}, \\ \mu_4\left(\exp i\mathfrak{a} \cdot_3 (x,X)\right) & = & \cosh i\mathfrak{a} \circ \mu_4\left((x,X)\right) \circ \cosh i\mathfrak{a} + \sinh i\mathfrak{a} \circ \mu_4\left((x,X)\right) \circ \sinh i\mathfrak{a} \\ & & + \cosh i\mathfrak{a} \circ \mu_3\left((x,X)\right) \circ \sinh i\mathfrak{a} + \sinh i\mathfrak{a} \circ \mu_3\left((x,X)\right) \circ \cosh i\mathfrak{a}. \end{array}$$

Therefore:

$$\begin{array}{ll}(\mu_3+\mu_4)\left(\exp i\mathfrak{a}\cdot_3(x,X)\right) & =\exp i\mathfrak{a}\circ(\mu_3+\mu_4)\circ\exp i\mathfrak{a},\\ (\mu_3-\mu_4)\left(\exp i\mathfrak{a}\cdot_3(x,X)\right) & =\exp(-i\mathfrak{a})\circ(\mu_3-\mu_4)\circ\exp(-i\mathfrak{a}),\end{array}$$

and

$$\left(\mu_4^2-\mu_3^2\right)\left(\exp i\mathfrak{a}\cdot_3(x,X)\right)=\exp(i\mathfrak{a})\circ\left(\mu_4^2\left((x,X)\right)-\mu_3^2\left((x,X)\right)\right)\circ\exp(-i\mathfrak{a}).$$

For $\mathfrak{a}\in\mathfrak{g}$ such that $\exp i\mathfrak{a}\cdot_3((x,X))=q_3((x,X))$, it follows that

$$\mu_4\left(q_3\left((x,X)\right)\right)=\exp(i\mathfrak{a})\circ\left(\mu_4^2\left((x,X)\right)-\mu_3^2\left((x,X)\right)\right)^{\frac{1}{2}}\circ\exp(-i\mathfrak{a}),$$

and

$$K_3\left((x,X)\right)=\frac{1}{4}\ \mathrm{Tr}\ \left(\left(\mu_4^2\left((x,X)\right)-\mu_3^2\left((x,X)\right)\right)^{\frac{1}{2}}-k^2\mathrm{Id}\right).$$

$\triangle$

Proposition 5.5 *:*
For every $(x,\,X)$ in $\mathcal{W}_k^{s_3}$, one has

$$K_3\left((x,X)\right)=\frac{k^2}{4}\ Tr\ \left(\left(\ Id\ +\frac{4}{k^4}\left(xX^*Xx^*-Xx^*Xx^*\right)\right)^{\frac{1}{2}}-\ Id\ \right).$$

$\square$ **Proof of Proposition** 5.5:
Since $\mathcal{W}_k^{s_3}\subset\mu_1^{-1}\left(-\frac{i}{2}k^2\ \mathrm{Tr}\ \right)\cap\mu_2^{-1}(0)$, for all (x,X) in $\mathcal{W}_k^{s_3}$, one has $x^*x-X^*X=k^2\ \mathrm{Id}$ and $X^*x=x^*X$. Hence :

$$\begin{array}{rcl}\left(\mu_4^2\left((x,X)\right)-\mu_3^2\left((x,X)\right)\right) & = & x^*xx^*x+x^*xX^*X+X^*Xx^*x+X^*XX^*X\\ & = & x^*x(X^*X+k^2)+x^*xX^*X+X^*Xx^*x\\ & & +(x^*x-k^2)X^*X-x^*Xx^*X\\ & & -x^*XX^*x-X^*xx^*X-X^*xX^*x\\ & = & k^4+4x^*xX^*X-4x^*Xx^*X.\end{array}$$

The result then follows after conjugation by x^{*-1} viewed as an operator of H_+ onto $\mathrm{Ran}\,x$. $\square$

Theorem 5.6 *:*
The symplectic form ω_3^{red} on the cotangent space $T'Gr_{res}^0$ admits a globally defined Kähler potential $\hat{K}_3$ on $T'Gr_{res}$, whose expression at $(P,\,V^*)\in T'Gr_{res}^0$ is given by

$$\begin{array}{rcl}\hat{K}_3\left((P,V^*)\right) & = & \frac{k^2}{4}\ Tr\ \left(\left(\ Id+4V^*V\right)^{\frac{1}{2}}-\ Id\right)\\ & = & \mathrm{g}_{Gr}\left(h\left(I_1R_{I_1V,V}\right)V,V\right),\end{array}$$

where :

$$h(u):=\frac{1}{u}(\sqrt{1+u}-1).$$

■ **Proof of Theorem** 5.6:
When (x, X) belongs to the level set, $x^*X = 0$ and

$$\begin{aligned} K_3\left((x, X)\right) &= \tfrac{k^2}{4}\ \mathrm{Tr}\ \Big(\big(\ \mathrm{Id}\ + \tfrac{4}{k^4}(xX^*Xx^*)\big)^{\frac{1}{2}} -\ \mathrm{Id}\ \Big) \\ &= \tfrac{k^2}{4}\ \mathrm{Tr}\ \Big(\big(\ \mathrm{Id}\ + 4V^*V\big)^{\frac{1}{2}} -\ \mathrm{Id}\ \Big), \end{aligned}$$

where $V = \frac{1}{k^2} X \circ x^*$. The theorem then follows from the identities:

$$\mathrm{g}_{Gr}\left(I_1 R_{I_1 V, V} V, V\right) = \frac{1}{4} \Re\ \mathrm{Tr}\ \left((4V^*V)^2\right)$$

and

$$\mathrm{g}_{Gr}\left((I_1 R_{I_1 V, V})^j V, V\right) = \frac{1}{4} \Re\ \mathrm{Tr}\ \left((4V^*V)^{j+1}\right),$$

and from the fact that $p_3^* \omega_3^{red} = dd^{c_3} K_3 = dd^{c_3} p_3^* \hat{K}_3 = p_3^* dd^{c_3} \hat{K}_3$ since p_3 is holomorphic with respect to the complex structure I_3. ■

5.5 The Kähler potential $\hat{K}_3$ of $\mathcal{O}^{\mathbb{C}}$ as a function of the characteristic angles

In this Subsection, we show that the general formulas of [6] giving K_3 in terms of the characteristic angles have an analogue in the infinite-dimensional setting. For this purpose, we use a section of the application ψ (defined in Theorem 5.2), which has been already used in the proof of Theorem 5.2.

Theorem 5.2 states that every ordered pair $(P,\, Q)$ belonging to $Gr_{res}^0 \times Gr_{res}^{*0}$ with $P \cap Q = \{0\}$ represents an element of the complexified orbit. A preimage $(x,\, X)$ of $(P,\, Q)$ by the application ψ : $\mathcal{W}_k^{s_3} \to \mathcal{O}^{\mathbb{C}}$ is given by :

$$\left\{ \begin{array}{l} x = k(\mathrm{Id}_P + \frac{1}{2}A) g_{|H_+} \\ X = -\frac{k}{2} A g_{|H_+}, \end{array} \right. \tag{9}$$

where A is a Hilbert-Schmidt operator from P to $P^\perp$ whose graph is $Q^\perp$ (determined modulo the right action of $GL(P)$), and where g is a unitary operator uniquely defined if P and Q are endowed with their canonical bases. Note that the eigenvalues $\{a_i^2\}_{i\in\mathbb{N}}$ of A^*A are independent of the operator A chosen to represent the ordered pair $(P,\, Q)$. If A is generic, i.e. if all the eigenvalues a_i^2 are distinct, it is possible to define pairs of characteristic lines $\{l_i, l_i'\}$, $i \in \mathbb{N}$, as follows. The complex line l_i is the eigenspace in P of the operator A^*A with respect to the eigenvalue a_i, and l_i' is the complex line in $Q^\perp$ which is the image of l_i under the operator $\mathrm{Id}_P + A$. The angle θ_i between the two complex line l_i and l_i' is defined by

$$\cos\theta_i = |\langle e_i, e_i' \rangle|,$$

where e_i is a unitary generator of l_i and where :

$$e'_i := \frac{e_i + A(e_i)}{|e_i + A(e_i)|}.$$

The angle θ_i is related to the eigenvalue a_i by the following formula :

$$\cos\theta_i = \frac{1}{\sqrt{1+a_i^2}}.$$

The latter expression makes sense even in the non-generic case, and allows one to uniquely define the set of characteristic angles $\theta_i \in (-\frac{\pi}{2}, +\frac{\pi}{2})$, $i \in \mathbb{N}$.

Remark 5.7 The orbit of an ordered pair (P, Q) in $Gr^0_{res} \times Gr^{0*}_{res}$ under the natural action of $GL_2(H)$ is characterized by the dimension of $P \cap Q$. The orbit of (P, Q) under the action of $\mathcal{U}_2(H)$ on $Gr^0_{res} \times Gr^{0*}_{res}$ is characterized by the set of characteristic angles θ_i.

Proposition 5.5 allows to express the Kähler potential K_3 on the complexified orbit either in terms of the eigenvalues a_i of A^*A or in terms of the characteristic angles θ_i :

Theorem 5.8 *The form ω_3^{red} defined on the complexified orbit $\mathcal{O}^{\mathbb{C}}$ and associated with the natural complex structure of $\mathcal{O}^{\mathbb{C}}$ satisfies $\omega_3^{red} = dd^{c_3}\hat{K}_3$ with :*

$$\hat{K}_3\left((P\,,\,Q)\right) = k^2\ Tr\ \left((\ Id_P + A^*A)^{\frac{1}{2}} -\ Id_P\right),$$

for $(P\,,\,Q)$ in $\mathcal{O}^{\mathbb{C}}$, where A is such that $Ran(Id_P{+}A) = Q^{\perp}$. Denoting by a_i the eigenvalues of the operator A^*A, and by θ_i the characteristic angles of the pair $(P\,,\,Q)$, one has :

$$\begin{aligned} \hat{K}_3\left((P\,,\,Q)\right) &= k^2 \textstyle\sum_{i\in\mathbb{N}} \left(\sqrt{1+a_i^2} - 1\right) \\ &= k^2 \textstyle\sum_{i\in\mathbb{N}} \left(\frac{1}{\cos\theta_i} - 1\right). \end{aligned}$$

■ **Proof of Theorem** 5.8:
From the proof of Theorem 5.5 it follows that the potential K_3 is given at an element (x, X) of the stable manifold $\mathcal{W}_k^{s_3}$ by :

$$K_3((x,X)) = \ \mathrm{Tr}\ \left(\left(k^4 + 4x^*xX^*X - 4x^*Xx^*X\right)^{\frac{1}{2}} - k^2\ \mathrm{Id}\right).$$

To proceed, let us recall the element (x, X) of $\mathcal{W}_k^{s_3}$ defined in the proof of Theorem 5.2 by :

$$\begin{aligned} x &= k(\mathrm{Id}_P + \tfrac{1}{2}A)\circ u_1 \\ X &= -\tfrac{k}{2}A\circ u_1, \end{aligned}$$

where u_1 is a unitary operator from H_+ to P. One has $\psi((x, X)) = (P, Q)$ and

$$\hat{K}_3((P, Q)) = K_3((x, X)) = k^2 \text{ Tr } \Big((\text{ Id}_P + u_1^* A^* A u_1)^{\frac{1}{2}} - \text{ Id}_P \Big),$$

which, after conjugation by u_1, gives the result. ■

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